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AN ELEMENTARY SUBSIDENCE MODEL

by Paul R. Paslay, P. E. # 44278, Manatee, Inc., F-4992

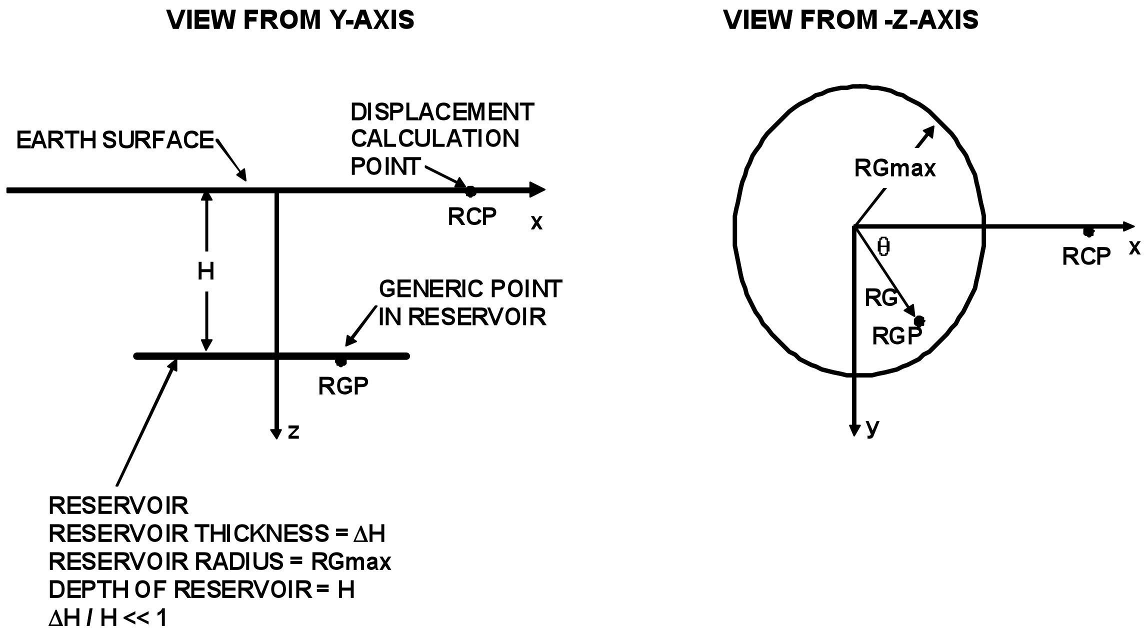
I. INTRODUCTION

The subsidence issue considered in this study is the prediction of the surface displacements, vertical and radial, owing to changes in the pore pressure within a reservoir. The model for the predictions is based on a poroelastic half-space containing a horizontal reservoir. The reservoir is composed of up to 25 circular, parallel layers each with a height that is small compared to its depth below the surface. The centers of the layers lie along a single vertical line. The pore pressure does not change outside the reservoir layers. The pore pressure change within each reservoir layer is composed of two parts. The first is a uniform pressure decrease in the reservoir from the original pressure distribution. The second is a steady-state pressure distribution derived from a specified drawdown pressure using a constant permeability Darcy's Law.

Each thin circular reservoir layer is modeled as a single layer of "pressure points". A pressure point is a small volume in an infinite poroelastic body that undergoes a pore pressure increase of Δp. The stress and displacement fields surrounding the pressure point are given in many elasticity textbooks (e.g. Reference 1). This pressure point solution is used to determine a Green's Function for constructing the solution for the half-space containing the reservoir. The tractions on the surface of the half-space are all zero.

A numerical integration is used for the circumferential and radial integrations that sum the contributions from the reservoir pressure points. These integrations yield predictions for the vertical and radial displacements.

II. DESCRIPTION OF ONE LAYER OF THE RESERVOIR CONFIGURATION



The sketch above shows the most important parameters for the configuration being studied. The components, Δx, Δy and Δz, of the vector from Point RGP in the reservoir to Point RCP on the x-axis are important for the subsidence calculations. They are,

$$\begin{aligned} \Delta x &= RCP - RG \cdot \cos(\vartheta) \\ \Delta y &= RG \cdot \sin(\vartheta) \\ \Delta z &= H \end{aligned} \qquad 1, 2, 3$$

Consequently, the distance, RGC, between Points RGP and RCP is,

$$RGC = RGC(RCP, RG, H, \vartheta) = \sqrt{\left(RCP - RG \cdot \cos(\vartheta)\right)^2 + \left(RG \cdot \sin(\vartheta)\right)^2 + H^2} \qquad 4$$

or, equivalently,

$$RGC = \sqrt{RCP^2 + RG^2 + H^2} \cdot \sqrt{1 - RD \cdot \cos(\vartheta)} \qquad 5$$

where,

$$RD = \frac{2 \cdot RCP \cdot RG}{RCP^2 + RG^2 + H^2} \qquad 6$$

Note that for this configuration the values of both RCP and RG are positive so that RD is always less than one.

Application of the pressure point solution, given below, to the points in the reservoir layer ($z = H$, $r = 0$ to RGmax and $\vartheta = 0$ to $2 \cdot \pi$) requires integration to find the displacement at RCP.

III. POINT PRESSURE SOLUTION

Reference 1, pp. 392-395, gives the solution for a spherical hole of radius a centrally located in a spherical, elastic body of radius b. The inner surface is subjected to a pressure of Δps with no shear traction. The tractions on the outer surface are zero. Let R be the radial coordinate in a spherical coordinate system centered in the body. The radial stress, σ_R, and tangential stress, σ_T, are given by,

$$\sigma_R = -\Delta ps\cdot\frac{a^3}{R^3}\cdot\frac{b^3 - R^3}{b^3 - a^3}$$
$$\sigma_T = \tfrac{1}{2}\cdot\Delta ps\cdot\frac{a^3}{R^3}\cdot\frac{b^3 + 2\cdot R^3}{b^3 - a^3} \qquad 7, 8$$

To obtain the solution for an infinite elastic medium with a hole of radius = a, let $b \to \infty$ in the two equations above so that,

$$\sigma_R = -\Delta ps\cdot\frac{a^3}{R^3}$$
$$\sigma_T = \tfrac{1}{2}\cdot\Delta ps\cdot\frac{a^3}{R^3} \qquad 9, 10$$

Clearly, for this solution the only displacement change is the radial displacement, Δu_{ps}, which is a function of r. The changes in the radial strain, $\Delta\varepsilon_R$, and tangential strain, $\Delta\varepsilon_T$, are,

$$\Delta\varepsilon_R = \frac{d\,\Delta u_{ps}}{dR} = \frac{1}{E}\cdot\left(\sigma_R - 2\cdot\nu\cdot\sigma_T\right) = -\frac{1+\nu}{E}\cdot\Delta ps\cdot\frac{a^3}{R^3}$$
$$\Delta\varepsilon_T = \frac{\Delta u_{ps}}{R} = \frac{1}{E}\cdot\left(\sigma_T - \nu\cdot(\sigma_R + \sigma_T)\right) = \frac{1+\nu}{2\cdot E}\cdot\Delta ps\cdot\frac{a^3}{R^3} \qquad 11, 12$$

where E is Young's Modulus of Elasticity and ν is Poisson's Ratio. Therefore, the solution for Δu_{ps} is,

$$\Delta u_{ps} = \frac{1+\nu}{2\cdot E}\cdot\Delta ps\cdot\frac{a^3}{R^2} \qquad 13$$

When points remote from the point pressure (R >> a) are considered the shape of the volume for the point pressure is not important so a^3 may be approximated by the volume, VOL, of the sphere (recall for a sphere the volume is $\frac{4}{3}\cdot\pi\cdot a^3$). The generalized result to be employed in the next section is,

$$\Delta u_{ps} = \frac{3\cdot(1+\nu)}{8\cdot\pi\cdot E}\cdot\Delta ps\cdot\frac{VOL}{R^2} \qquad 14$$

In order to complete the point pressure solution the solution for the displacements is required. This second solution is for a solid, elastic sphere of radius a subjected to a uniform external pressure and in addition an internal volumetric strain of $3\cdot\Delta\varepsilon$. Let the external pressure be designated as Δpx and the radial displacement as Δu_x. The solution for this almost trivial problem is,

$$\Delta u_x = R\cdot\left(\Delta\varepsilon^* - (1-2\cdot\nu)\cdot\frac{\Delta px}{E}\right) \qquad 15$$

The nominal radius of this sphere is a. Now requiring radial force equilibrium for the two solutions at R = a so that,

$$\Delta ps = \Delta px \qquad 16$$

and requiring displacement continuity at R = a so that,

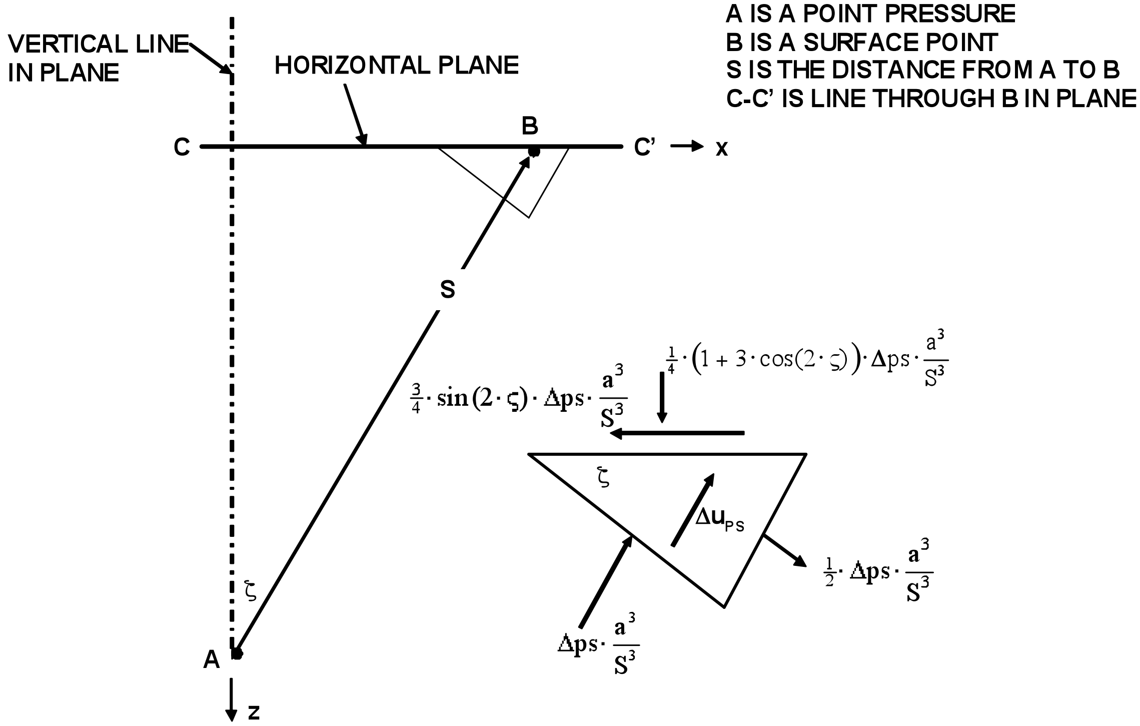
$$\Delta u_{ps} = \Delta u_x \qquad 17$$

gives the desired result for the point pressure solution that,

$$\Delta ps = \frac{2\cdot E}{3\cdot(1-\nu)}\cdot\Delta\varepsilon^* \qquad 18$$

This last relation between Δps and $\Delta\varepsilon^*$ is used when the loading changes are given in terms of $\Delta\varepsilon^*$ rather than Δps. For example, in thermoelastic analysis $\Delta\varepsilon^* = \alpha\cdot\Delta T$ where α is the coefficient of linear thermal expansion and ΔT is the increase in temperature. Reference 2, Chapter 1 refers to $\Delta\varepsilon^*$ as an "eigenstrain" and uses this concept to determine solutions for many problems. The subsidence problem for an ellipsoidal reservoir with a uniform pore pressure reduction is included in Reference 2.

Finally, for this point pressure solution consider two points in a single vertical plane as shown below. The results given above are used to find changes of the radial displacement and stresses at the horizontal surface, Point B, when a point pressure occurs at Point A below the surface. Now, as a thought experiment, consider rotating the vertical plane in the figure below between -180° and +180° about Line C-C'. The σ_R and σ_T stresses at the surface will not be altered but the surface stresses will vary with rotation angle. In particular, a surface shear stress is developed that is perpendicular to Line C-C'. Note that this shear stress is an odd function of the angle of rotation. Consequently, a fictitious point pressure can be superposed at Point D with a negative point pressure equal in magnitude to the one at a generic point so that the surface tractions will all be zero. If the location of the generic point is A' is (x, y, z) then the location of Point D is (x, y, -z). All points in the reservoir are matched in this way in the analysis in the next section.



IV. THE HALF-SPACE PROBLEM

The boundary of the half-space in the model considered here has no applied nontrivial tractions. The only loading in the half-space is from a finite number of point pressures applied in the horizontal, thin, circular reservoir layers. The point pressure solutions for an infinite medium cause nontrivial stresses on the surface of the half-space. The procedure described in the previous section is used to remove the surface tractions by superposing fictitious, negative point pressures above the surface. Each point pressure causes a displacement vector, $\vec{u}$, at the surface that is parallel to the line joining the point pressure location and the surface point being considered. The unit length vector parallel to this line and pointed toward the surface is denoted by $\vec{V}$. Taking into account a single point pressure with Δp pressure increase gives,

$$\vec{u} = \frac{3 \cdot (1 + \nu)}{8 \cdot \pi} \cdot \frac{\Delta p}{E} \cdot \frac{VOL}{S^2} \cdot \vec{V} \qquad 19$$

Now using Equations 1 through 4 the components of $\vec{u}$ are found with obvious notation as,

$$\begin{aligned} u_x &= \frac{\Delta x}{S} \cdot |\vec{u}| \\ u_y &= \frac{\Delta y}{S} \cdot |\vec{u}| \\ u_z &= \frac{\Delta z}{S} \cdot |\vec{u}| \end{aligned} \qquad 20, 21, 22$$

When the displacement changes of the corresponding fictitious, negative point pressure are added to the displacement changes given by Equations 20, 21 and 22 the result is,

$$\begin{aligned} u_x &= 2 \cdot \frac{\Delta x}{S} \cdot |\vec{u}| \\ u_y &= 0 \\ u_z &= 2 \cdot \frac{\Delta z}{S} \cdot |\vec{u}| \end{aligned} \qquad 23, 24, 25$$

Equations 23, 24 and 25 are used in the calculation results presented in this study.

V. SOME DETAILS CONCERNING THE NUMERICAL INTEGRATION

Program SUBSI6 was prepared as a FORTRAN computer program to make the calculations described above. It has two straightforward numerical integrations. The first integration is on the angle ϑ and the second integration is on the cylindrical radius r. The integrations find the outward radial and upward vertical displacements. These determinations are made for each of the selected radial positions. The integration on ϑ extends from 0 to π and is then multiplied by two. The radial integration extends from the reservoir borehole radius to the radial boundary of the reservoir. When the coordinates for the RG stations have been set, RG(I), in the program, the point pressure values, P(I), are determined in source code notation as follows,

$$P(I) = PO - POR + (PI - PO)\cdot\frac{\ln\left(\frac{RGMAX}{RG(I)}\right)}{\ln(DRAT)} \qquad 26$$

The input data required for a computer run are (using the notation in the program source code),

RCMAX	= maximum calculation radius, in
NC	= number of spaces for RCMAX (< 51)
DRAT	= (RCMAX) / (borehole radius)
E	= Young's modulus, psi
NU	= Poisson's ratio
NH	= number of layers

A tabulation of layer properties one row for each layer
:

RGMAX NG H DH PI PO POR

where for the Ith layer,

RGMAX(I)	= reservoir radius, in
NG(I)	= number of spaces for RG integration
H(I)	= mean depth of reservoir below surface, in
DH(I)	= vertical thickness of reservoir, in
PI(I)	= producing pressure at reservoir borehole, psi
PO(I)	= reservoir pressure at RGMAX, psi
POR(I)	= original reservoir uniform pressure, psi

The output data file contains all of the input data as well as a tabulation for I = 1 to NC+1 of,

RC(I)	= radial position, in
DUV(I)	= upward vertical component of surface displacement change, in

DUR(I) = outward radial component of surface displacement change, in

and,

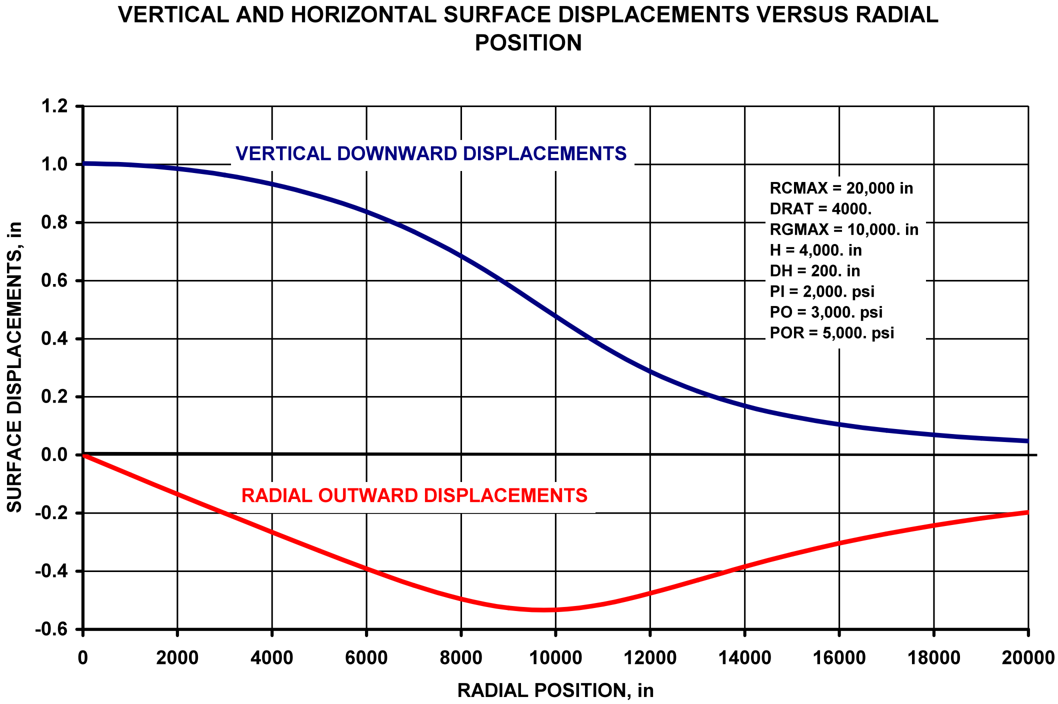
VOLUME OF SURFACE SUBSIDENCE, cu.in.

VI. ILLUSTRATIVE EXAMPLES

The input data for the first illustrative example are,

RCMAX	= 20,000. in					
NC	= 20					
DRAT	= 4,000					
E	= 500,000. psi					
NU	= 0.25					
NH	= 1					
10,000.	100	4000.	200.	2,000.	3,000.	5,000.
RGMAX	NG	H	DH	PI	PO	POR

Note that the reservoir has only one layer. The results given below were checked by an altered, slower version of the program that uses a Simpson's Rule integration scheme to determine successive values for the RG integration as NG is increased. The progression stops when the change in the integral between steps becomes smaller than a preset value. The results between the programs agreed with differences in the vertical displacements occurring in the fourth significant digit and for the radial displacements in the third significant digit.



A result of interest in some calculations is the surface subsidence volume displaced. For the case shown in the above figure,

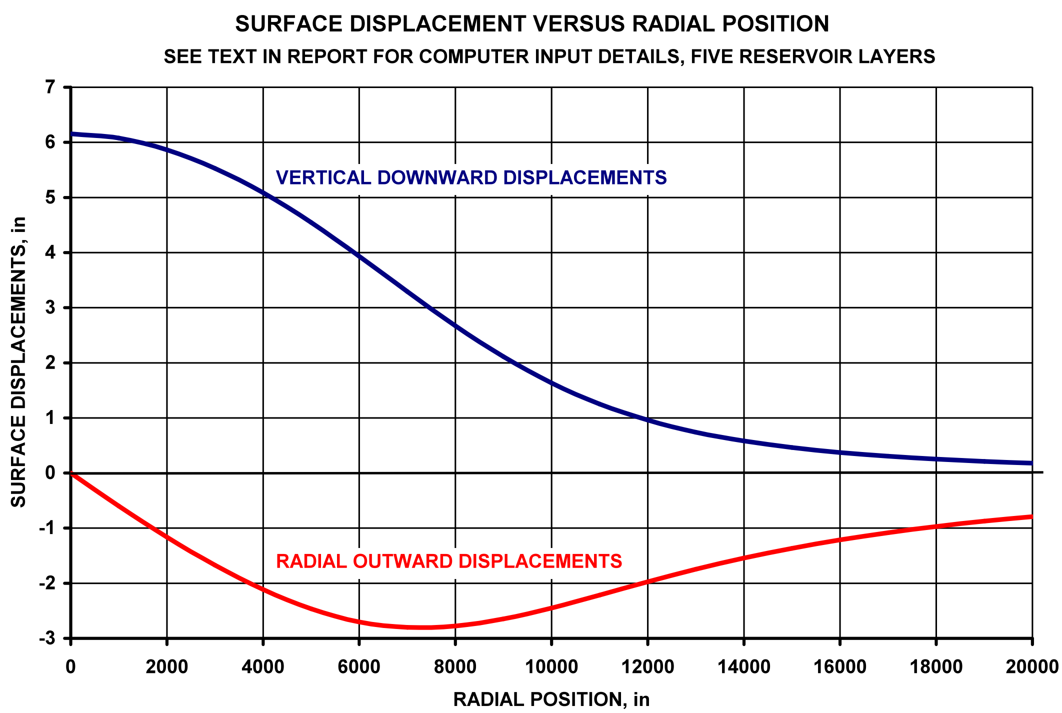
Displaced volume = 4.12E8 cu.in.

The input data file for the second illustrative problem is given below. The data show that there are five layers in the reservoir.

20000.D0	= RCMAX	= maximum calculation radius, in
20	= NC	= number of spaces for RCMAX (< 51)
4000.D0	= DRAT	= (RCMAX) / (borehole radius)
5.D5	= E	= Young's modulus, psi
0.25D0	= NU	= Poisson's ratio
5	= NH	= number of layers

RGMAX	NG	H	DH	PI	PO	POR
6000.D0	100	3000.D0	200.D0	2000.D0	3000.D0	3500.D0
8000.D0	100	3500.D0	200.D0	2250.D0	3250.D0	3750.D0
10000.D0	100	4000.D0	200.D0	2500.D0	3500.D0	4000.D0
8000.D0	100	4500.D0	200.D0	2750.D0	3750.D0	4250.D0
6000.D0	100	5000.D0	200.D0	3000.D0	4000.D0	4500.D0

The figure below gives the surface displacements for this illustrative problem.



Displaced volume = 1.74E9 cu.in.

VII. REFERENCES

1. S. P. Timoshenko and J. N. Goodier, *Theory of Elasticity*, Third Edition, McGraw Hill Book Company, 1970
2. Toshio Mura, *Micromechanics of Defects in Solids*, Second, Revised Edition, Kluwer Academic Publishers, 1987
3. J. Geertsma, "Land Subsidence Above Compacting Oil and Gas Reservoirs," Journal of Petroleum Technology, SPE 3730, June, 1973, pp. 734-744
4. J. Geertsma, "Problems of Rock Mechanics in Petroleum Production Engineering," Proceedings of the First Congress of the International Society of Rock Mechanics, Lisbon, 1966, pp. 585-594

ADDENDUM A – A NOTE ON THE ANGULAR INTEGRATION

The original program that was prepared to obtain data for this study used numerical integration in both the radial and angular directions. This program has been altered by replacing the angular integration with complete Legendre elliptic integrals, K(k) and E(k), as described below.

Equations 19 through 22 may be expressed as,

$$\overline{u}_x = \frac{\Delta x}{RGC} \cdot |\vec{u}| = \frac{A}{\left(1 - RD \cdot \cos(\vartheta)\right)^{1.5}} + \frac{B \cdot \cos(\vartheta)}{\left(1 - RD \cdot \cos(\vartheta)\right)^{1.5}} \qquad \text{A1}$$

$$\overline{u}_y = \frac{\Delta y}{RGC} \cdot |\vec{u}| = \frac{C \cdot \sin(\vartheta)}{\left(1 - RD \cdot \cos(\vartheta)\right)^{1.5}} \qquad \text{A2}$$

$$\overline{u}_z = \frac{\Delta z}{RGC} \cdot |\vec{u}| = \frac{D}{\left(1 - RD \cdot \cos(\vartheta)\right)^{1.5}} \qquad \text{A3}$$

with RD, A, B, C and D independent of ϑ. Let,

$$AA \equiv \frac{3 \cdot (1 + \nu)}{8 \cdot \pi} \cdot \frac{\Delta p \cdot VOL}{E \cdot \left(RCP^2 + RG^2 + H^2\right)^{1.5}} \qquad \text{A4}$$

then,

$$RD = \frac{2 \cdot RCP \cdot RG}{RCP^2 + RG^2 + H^2}, \qquad \text{for this problem } 0 \le RD \le 1 \qquad \text{A5}$$

$$A = AA \cdot RCP \qquad \text{A6}$$

$$B = -AA \cdot RG \qquad \text{A7}$$

$$C = AA \cdot RG \qquad \text{A8}$$

$$D = AA \cdot H \qquad \text{A9}$$

Consequently, the integration on ϑ can be accomplished when the following integrals have been evaluated.

$$I1 = \int_0^{\pi} \left(1 - RD \cdot \cos(\vartheta)\right)^{-1.5} \cdot d\vartheta \qquad \text{A10}$$

$$I2 = \int_0^{\pi} \cos(\vartheta) \cdot (1 - RD \cdot \cos(\vartheta))^{-1.5} \cdot d\vartheta \qquad A11$$

$$I3 = \int_0^{\pi} \sin(\vartheta) \cdot (1 - RD \cdot \cos(\vartheta))^{-1.5} \cdot d\vartheta \qquad A12$$

The integrations yield,

$$I1 = \frac{2}{(1+RD)^{1.5}} \cdot \sqrt{\frac{1+RD}{1-RD}} \cdot E\left(\sqrt{\frac{2 \cdot RD}{RD-1}}\right) \qquad A13$$

$$I2 = \frac{2}{RD \cdot (1+RD)^{1.5}} \cdot \sqrt{\frac{1+RD}{1-RD}} \cdot \left(E\left(\sqrt{\frac{2 \cdot RD}{RD-1}}\right) - (1+RD) \cdot K\left(\sqrt{\frac{2 \cdot RD}{RD-1}}\right)\right) \qquad A14$$

$$I3 = \frac{2}{RD} \cdot \left(\frac{1}{\sqrt{1-RD}} - \frac{1}{\sqrt{1+RD}}\right) \qquad A15$$

where,

$$K(k) = \text{complete elliptic integral of } 1^{st} \text{ kind} = \int_0^{\frac{\pi}{2}} \frac{d\varphi}{\sqrt{1 - k^2 \cdot \sin^2(\varphi)}} \qquad A16$$

$$E(k) = \text{complete elliptic integral of } 2^{nd} \text{ kind} = \int_0^{\frac{\pi}{2}} \sqrt{1 - k^2 \cdot \sin^2(\varphi)} \cdot d\varphi \qquad A17$$

Subroutines for K(k) and E(k) are readily available (e.g. – Numerical Recipes). Note that the argument k is an imaginary number while K(k) and E(k) are real.

When the integrals in Equations A10, A11 and A12 are extended from π to $2 \cdot \pi$ and the presence of the fictitious negative point pressure at -z are taken into account, analogous to Equations 23, 24 and 25, the results are that,

u_x = x component of displacement = $4 \cdot \bar{u}_x$ A18

u_y = y component of displacement = 0 A19

u_z = z component of displacement = $4 \cdot \bar{u}_z$ A20

ADDENDUM B – DISPLACEMENTS BELOW THE HALF-SPACE SURFACE

The procedure described above calculates the surface displacements by applying pressure points within the reservoir being studied. In order to have surface tractions (normal and shear stresses) vanish, a fictitious pressure point is superposed on each pressure point applied in the reservoir. That is, for a pressure point in the reservoir at (r, ϑ, H) there is a pressure point of equal magnitude and opposite sign applied at the fictitious point $(r, \vartheta, -H)$.

The displacements at points below the surface may be determined using a similar procedure. For a generic point (r, 0, z) with $0 < z < H$, and a pressure point (with Δp and VOL) at $\left(RG \cdot \cos(\vartheta), RG \cdot \sin(\vartheta), H\right)$, the corresponding fictitious pressure point is located at $\left(RG \cdot \cos(\vartheta), RG \cdot \sin(\vartheta), -H\right)$. Let 1 be the subscript for the pressure point in the reservoir and 2 be the subscript for the fictitious pressure point. Now, referring to Equations 20, 21 and 22,

$$\Delta x_1 = r - RG \cdot \cos(\vartheta)$$
$$\Delta y_1 = -RG \cdot \sin(\vartheta)$$
$$\Delta z_1 = z - H$$
$$RR_1^2 = r^2 + RG^2 + (z - H)^2 \qquad \text{B1 – B6}$$
$$RDR_1 = \frac{2 \cdot r \cdot RG}{RR_1^2}$$
$$S_1^2 = RR_1^2 \cdot \left(1 - RDR_1 \cdot \cos(\vartheta)\right)$$

$$\Delta x_2 = r - RG \cdot \cos(\vartheta)$$
$$\Delta y_2 = -RG \cdot \sin(\vartheta)$$
$$\Delta z_2 = z + H$$
$$RR_2^2 = r^2 + RG^2 + (z + H)^2 \qquad \text{B7 – B12}$$
$$RDR_2 = \frac{2 \cdot r \cdot RG}{RR_2^2}$$
$$S_2^2 = RR_2^2 \cdot \left(1 - RDR_2 \cdot \cos(\vartheta)\right)$$

When Equation 19 is taken into account and U is defined as,

$$U = \frac{3 \cdot (1 + \nu)}{8 \cdot \pi} \cdot \frac{\Delta p}{E} \cdot VOL \qquad \text{B13}$$

The displacement at (r, 0, z) is given by,

$$u_x = \left(\frac{\Delta x_1}{S_1^{1.5}} - \frac{\Delta x_2}{S_2^{1.5}} \right) \cdot U \qquad \text{B14}$$

$$u_y = \left(\frac{\Delta y_1}{S_1^{1.5}} - \frac{\Delta y_2}{S_2^{1.5}} \right) \cdot U \qquad \text{B15}$$

$$u_z = \left(\frac{\Delta z_1}{S_1^{1.5}} - \frac{\Delta z_2}{S_2^{1.5}} \right) \cdot U \qquad \text{B16}$$

The above equations in this Addendum show that the displacement components corresponding to a specific value of RG may all be expressed in the form,

$$\begin{aligned} u_i = &\frac{A_{1i}}{\left(1 - RD_{1i} \cdot \cos(\vartheta)\right)^{1.5}} + \frac{B_{1i} \cdot \cos(\vartheta)}{\left(1 - RD_{1i} \cdot \cos(\vartheta)\right)^{1.5}} \\ &+ \frac{A_{2i}}{\left(1 - RD_{2i} \cdot \cos(\vartheta)\right)^{1.5}} + \frac{B_{2i} \cdot \cos(\vartheta)}{\left(1 - RD_{2i} \cdot \cos(\vartheta)\right)^{1.5}} \qquad i = x, y, z \end{aligned} \qquad \text{B17}$$

Consequently, the integrals given in Equations A10, A11 and A12 may be applied to determine the displacements at points below the surface.

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