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THE LUBINSKI-WOODS PROBLEM USING FORCE EQUILIBRIUM RATHER THAN POTENTIAL ENERGY

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Consider a helical elastica of pitch PIT and radius R. Let the axis of the helix coincide with the z-axis of a Cartesian, right-handed, xyz coordinate system. Let s measure length along the helix so that locations on the helix may be identified by s. The vector $\vec{r}$ from the origin of the coordinate system to any point on the helix may be written as,

$$\vec{r} = \vec{i} \cdot R \cdot \cos\vartheta + \vec{j} \cdot R \cdot \sin\vartheta + \vec{k} \cdot s \cdot \cos\alpha$$

where,

$$\tan\alpha = \frac{2 \cdot \pi \cdot R}{PIT}$$

$$\vartheta = 2 \cdot \pi \cdot \frac{\cos\alpha}{PIT} \cdot s$$

$$R \cdot \frac{d\vartheta}{ds} = \sin\alpha$$

The angles α and ϑ may be called the helix angle and azimuth angle, respectively. The unit length vector tangent to the helix is,

$$\vec{\tau} \equiv \frac{d\vec{r}}{ds} = -\vec{i} \cdot R \cdot \sin\vartheta \cdot \frac{d\vartheta}{ds} + \vec{j} \cdot R \cdot \cos\vartheta \cdot \frac{d\vartheta}{ds} + \vec{k} \cdot \cos\alpha = -\vec{i} \cdot \sin\alpha \cdot \sin\vartheta + \vec{j} \cdot \sin\alpha \cdot \cos\vartheta + \vec{k} \cdot \cos\alpha$$

The unit length radial vector is given by,

$$\vec{\rho} \equiv \vec{i} \cdot \cos\vartheta + \vec{j} \cdot \sin\vartheta$$

A third unit length vector is now defined that is mutually perpendicular to $\vec{\tau}$ and $\vec{\rho}$ and forms a $\vec{\tau}, \vec{\rho}, \vec{\eta}$ right handed system,

$$\vec{\eta} = \vec{\tau} \times \vec{\rho} = -\vec{i} \cdot \sin\vartheta \cdot \cos\alpha + \vec{j} \cdot \cos\vartheta \cdot \cos\alpha - \vec{k} \cdot R \cdot \frac{d\vartheta}{ds} = -\vec{i} \cdot \cos\alpha \cdot \sin\vartheta + \vec{j} \cdot \cos\alpha \cdot \cos\vartheta - \vec{k} \cdot \sin\alpha$$

Finally, the curvature vector of the helix, $\vec{\kappa}$, is given by,

$$\vec{\kappa} = \frac{d^2\vec{r}}{ds^2} = -R \cdot \left(\frac{d\vartheta}{ds}\right)^2 \cdot \left(\vec{i} \cdot \cos\vartheta + \vec{j} \cdot \sin\vartheta\right) = -\frac{\sin^2\alpha}{R} \cdot \left(\vec{i} \cdot \cos\vartheta + \vec{j} \cdot \sin\vartheta\right) = -\frac{\sin^2\alpha}{R} \cdot \vec{\rho}$$

so that,

$$|\vec{\kappa}| = R \cdot \left(\frac{d\vartheta}{ds}\right)^2 = \frac{\sin^2\alpha}{R}$$

The three base vectors defined above are used in the analysis below. It is, therefore, helpful to invert this set to obtain,

$$\vec{i} = -\vec{\tau} \cdot R \cdot \frac{d\vartheta}{ds} \cdot \sin\vartheta + \vec{\rho} \cdot \cos\vartheta - \vec{\eta} \cdot \cos\alpha \cdot \sin\vartheta = -\vec{\tau} \cdot \sin\alpha \cdot \sin\vartheta + \vec{\rho} \cdot \cos\vartheta - \vec{\eta} \cdot \cos\alpha \cdot \sin\vartheta$$
$$\vec{j} = \vec{\tau} \cdot R \cdot \frac{d\vartheta}{ds} \cdot \cos\vartheta + \vec{\rho} \cdot \sin\vartheta + \vec{\eta} \cdot \cos\alpha \cdot \cos\vartheta = \vec{\tau} \cdot \sin\alpha \cdot \cos\vartheta + \vec{\rho} \cdot \sin\vartheta + \vec{\eta} \cdot \cos\alpha \cdot \cos\vartheta$$
$$\vec{k} = \vec{\tau} \cdot \cos\alpha - \vec{\eta} \cdot R \cdot \frac{d\vartheta}{ds} = \vec{\tau} \cdot \cos\alpha - \vec{\eta} \cdot \sin\alpha$$

noting that,

$$|\vec{r}| = \sqrt{s^2 \cdot \cos^2\alpha + R^2}$$
$$\vec{\tau} \times \vec{\rho} = \vec{\eta}$$
$$\vec{\rho} \times \vec{\eta} = \vec{\tau}$$
$$\vec{\eta} \times \vec{\tau} = \vec{\rho}$$

as well as the derivatives,

$$\frac{d\vec{\tau}}{ds} = -R \cdot \left(\frac{d\vartheta}{ds}\right)^2 \cdot \vec{\rho} = -\frac{\sin^2\alpha}{R} \cdot \vec{\rho}$$
$$\frac{d\vec{\rho}}{ds} = R \cdot \left(\frac{d\vartheta}{ds}\right)^2 \cdot \vec{\tau} + \frac{d\vartheta}{ds} \cdot \cos\alpha \cdot \vec{\eta} = \frac{\sin^2\alpha}{R} \cdot \vec{\tau} + \frac{\sin\alpha \cdot \cos\alpha}{R} \cdot \vec{\eta}$$
$$\frac{d\vec{\eta}}{ds} = -\cos\alpha \cdot \frac{d\vartheta}{ds} \cdot \vec{\rho} = -\frac{\cos\alpha \cdot \sin\alpha}{R} \cdot \vec{\rho}$$

The external loading on the helix, w, is a uniform radial inward load per unit helix length. In general, a force and moment will be required on the cross section of the elastica. This force, $\vec{F}$, is resolved using the three mutually orthogonal base vectors defined above so that,

$$\vec{F} = \vec{\tau} \cdot F_\tau + \vec{\rho} \cdot F_\rho + \vec{\eta} \cdot F_\eta$$

where each component is independent of s. In order to determine if this can be a solution, the force must be substituted into the force and moment equilibrium equations and they must be satisfied. The force equilibrium equation is,

$$\frac{d\vec{F}}{ds} = w \cdot \left(\vec{i} \cdot \cos\vartheta + \vec{j} \cdot \sin\vartheta\right) = w \cdot \vec{\rho}$$

When the components are required to separately vanish, the result is,

$$F_\rho = 0$$

$$w = -F_\tau \cdot R \cdot \left(\frac{d\vartheta}{ds}\right)^2 - F_\eta \cdot \cos\alpha \cdot \frac{d\vartheta}{ds} = -\frac{\sin^2\alpha}{R} \cdot F_\tau - \frac{\sin\alpha \cdot \cos\alpha}{R} \cdot F_\eta$$

Note that the force equilibrium condition yields only two scalar equations. These two conditions must be satisfied in order to satisfy force equilibrium. The moment equilibrium condition is,

$$\frac{d\vec{M}}{ds} = \vec{F} \times \vec{\tau} = -\vec{\eta} \cdot F_\rho + \vec{\rho} \cdot F_\eta = \vec{\rho} \cdot F_\eta$$

where,

$$\vec{M} = \vec{\tau} \cdot M_\tau + \vec{\rho} \cdot M_\rho + \vec{\eta} \cdot M_\eta$$

and the moment components are independent of s. The moment equilibrium condition leads to,

$$M_\rho = 0$$

$$F_\eta = -R \cdot \left(\frac{d\vartheta}{ds}\right)^2 \cdot M_\tau - \cos\alpha \cdot \frac{d\vartheta}{ds} \cdot M_\eta = -\frac{\sin^2\alpha}{R} \cdot M_\tau - \frac{\sin\alpha \cdot \cos\alpha}{R} \cdot M_\eta$$

The elastic stress-strain equations for this problem are,

$$\vec{M} \times \frac{d\vec{r}}{ds} = \vec{M} \times \vec{\tau} = -\vec{\eta} \cdot M_\rho + \vec{\rho} \cdot M_\eta = \vec{\rho} \cdot M_\eta = EI \cdot \frac{d^2\vec{r}}{ds^2} = -\vec{\rho} \cdot EI \cdot R \cdot \left(\frac{d\vartheta}{ds}\right)^2 = -\frac{EI \cdot \sin^2\alpha}{R} \cdot \vec{\rho}\ ,$$

$$M_\eta = -EI \cdot R \cdot \left(\frac{d\vartheta}{ds}\right)^2 = EI \cdot \frac{d^2\vec{r}}{ds^2} = -\frac{EI \cdot \sin^2\alpha}{R}$$

and

$$\vec{M} \bullet \frac{d\vec{r}}{ds} = \vec{M} \bullet \vec{\tau} = M_\tau = GI_P \cdot \frac{d}{ds}\left(\varphi - 2 \cdot \pi \cdot \cos\alpha \cdot \frac{s}{PIT}\right) = GI_P \cdot \left(\frac{d\varphi}{ds} - \frac{\sin\alpha}{R}\right) = GI_P \cdot \left(\frac{d\varphi}{ds} - \frac{d\vartheta}{ds}\right)$$

where $\frac{d\varphi}{ds} - \frac{d\vartheta}{ds}$ is the elastic twist per unit length, EI is the bending stiffness and GI_P is the torsional stiffness. These equations reduce to,

$$M_\rho = 0$$

$$M_\tau = GI_P \cdot \left(\frac{d\varphi}{ds} - \frac{d\vartheta}{ds}\right) = GI_P \cdot \left(\frac{d\varphi}{ds} - \frac{2 \cdot \pi \cdot \cos\alpha}{PIT}\right) = GI_P \cdot \left(\frac{d\varphi}{ds} - \frac{\sin\alpha}{R}\right)$$

$$M_\eta = -EI \cdot R \cdot \left(\frac{d\vartheta}{ds}\right)^2 = -\frac{EI \cdot \sin^2\alpha}{R}$$

To summarize, in order to satisfy equilibrium, the following equations must be satisfied,

$$F_\rho = 0$$

$$w = -F_\tau \cdot R \cdot \left(\frac{d\vartheta}{ds}\right)^2 - F_\eta \cdot \cos\alpha \cdot \frac{d\vartheta}{ds} = -\frac{\sin^2\alpha}{R} \cdot F_\tau - \frac{\cos\alpha \cdot \sin\alpha}{R} \cdot F_\eta$$

$$M_\rho = 0$$

$$F_\eta = -R \cdot \left(\frac{d\vartheta}{ds}\right)^2 \cdot M_\tau - \cos\alpha \cdot \frac{d\vartheta}{ds} \cdot M_\eta = -\frac{\sin^2\alpha}{R} \cdot M_\tau - \frac{\cos\alpha \cdot \sin\alpha}{R} \cdot M_\eta$$

$$M_\tau = GI_P \cdot \left(\frac{d\varphi}{ds} - \frac{d\vartheta}{ds}\right) = GI_P \cdot \left(\frac{d\varphi}{ds} - \frac{\sin\alpha}{R}\right)$$

$$M_\eta = -EI \cdot R \cdot \left(\frac{d\vartheta}{ds}\right)^2 = -\frac{EI \cdot \sin^2\alpha}{R}$$

Using the above set, the fourth and second of the above equations can be written as,

$$\frac{R^2 \cdot F_\eta}{EI} = \sin^3\alpha \cdot \cos\alpha - \sin^2\alpha \cdot \frac{GI_P \cdot R}{EI} \cdot \left(\frac{d\varphi}{ds} - \frac{\sin\alpha}{R}\right)$$

$$\begin{aligned}\frac{R^3 \cdot w}{EI} &= -\frac{R^2 \cdot F_\tau}{EI} \cdot \sin^2\alpha - \sin^4\alpha \cdot \cos^2\alpha + \sin^3\alpha \cdot \cos\alpha \cdot \frac{GI_P \cdot R}{EI} \cdot \left(\frac{d\varphi}{ds} - \frac{\sin\alpha}{R}\right) \\ &= -\frac{R^2 \cdot F_\tau}{EI} \cdot \sin^2\alpha - \sin^4\alpha \cdot \cos^2\alpha + \sin^3\alpha \cdot \cos\alpha \cdot \frac{R \cdot M_\tau}{EI}\end{aligned}$$

It is interesting to note that, for constant F_τ and M_τ,

$$\begin{aligned}\frac{d}{d\alpha}\left(\frac{R^3 \cdot w}{EI}\right) &= -2 \cdot \frac{R^2 \cdot F_\tau}{EI} \cdot \sin\alpha \cdot \cos\alpha - 4 \cdot \sin^3\alpha \cdot \cos^3\alpha + 2 \cdot \sin^5\alpha \cdot \cos\alpha \\ &\quad + \frac{R \cdot M_\tau}{EI} \cdot \left(3 \cdot \sin^2\alpha \cdot \cos^2\alpha - \sin^4\alpha\right)\end{aligned}$$

In the second-to-last equation the loads, w and F_τ, and changes of configuration, α and $\frac{d\varphi}{ds}$, appear. A typical problem is that F_τ and $\frac{d\varphi}{ds}$ (or M_τ) are specified. This specified loading is not adequate for the determination of a unique solution.

The resolved force, F_{AXIAL}, and moment, M_{AXIAL}, along the helix axis at the same elevation are given by,

$$F_{AXIAL} = F_\tau \cdot \cos\alpha - F_\eta \cdot \sin\alpha$$

$$\vec{M}_{AXIAL} = \left(M_\tau + R \cdot F_\eta\right) \cdot \cos\alpha - \left(M_\eta - R \cdot F_\tau\right) \cdot \sin\alpha$$

The resolved force, $F_{LATERAL}$, and moment, $M_{LATERAL}$, perpendicular to the helix axis at the same elevation are,

$$F_{LATERAL} = F_\tau \cdot \sin\alpha + F_\eta \cdot \cos\alpha$$

$$\vec{M}_{LATERAL} = \left(M_\eta - R \cdot F_\tau\right) \cdot \cos\alpha + \left(M_\tau + R \cdot F_\eta\right) \cdot \sin\alpha$$

A convenient dimensionless form for these equations is given below,

$$\frac{w \cdot R^3}{EI \cdot \sin^4\alpha} = -\frac{F_\tau \cdot R^2}{EI \cdot \sin^2\alpha} - \cos^2\alpha + \frac{\cos\alpha}{\sin\alpha} \cdot \frac{M_\tau \cdot R}{EI}$$

$$\frac{F_\tau \cdot R^2}{EI \cdot \sin^2\alpha} = -\frac{w \cdot R^3}{EI \cdot \sin^4\alpha} - \cos^2\alpha + \frac{\cos\alpha}{\sin\alpha} \cdot \frac{M_\tau \cdot R}{EI}$$

$$\frac{F_\eta \cdot R^2}{EI \cdot \sin^2\alpha} = +\sin\alpha \cdot \cos\alpha - \frac{M_\tau \cdot R}{EI}$$

$$\frac{F_\eta}{F_\tau} = \frac{\sin\alpha \cdot \cos\alpha - \dfrac{M_\tau \cdot R}{EI}}{-\dfrac{w \cdot R^3}{EI \cdot \sin^4\alpha} - \cos^2\alpha + \dfrac{M_\tau \cdot R}{EI}}$$

$$\frac{M_\tau \cdot R}{EI} = \frac{GI_P}{EI} \cdot \left(R \cdot \frac{d\varphi}{ds} - \sin\alpha\right)$$

$$\frac{M_\eta \cdot R}{EI} = -\sin^2\alpha$$

$$\frac{F_{AXIAL} \cdot R^2}{EI \cdot \sin^2\alpha} = \frac{F_\tau \cdot R^2 \cdot \cos\alpha}{EI \cdot \sin^2\alpha} - \frac{F_\eta \cdot R^2 \cdot \sin\alpha}{EI \cdot \sin^2\alpha}$$

$$\frac{M_{AXIAL} \cdot R}{EI} = \left(\frac{M_\tau \cdot R}{EI} + \sin^2\alpha \cdot \frac{F_\eta \cdot R^2}{EI \cdot \sin^2\alpha}\right) \cdot \cos\alpha + \left(-\frac{M_\eta \cdot R}{EI} + \sin^2\alpha \cdot \frac{F_\tau \cdot R^2}{EI \cdot \sin^2\alpha}\right) \cdot \sin\alpha$$

$$\frac{F_{LATERAL} \cdot R^2}{EI \cdot \sin^2\alpha} = \frac{F_\tau \cdot R^2 \cdot \sin\alpha}{EI \cdot \sin^2\alpha} + \frac{F_\eta \cdot R^2 \cdot \cos\alpha}{EI \cdot \sin^2\alpha}$$

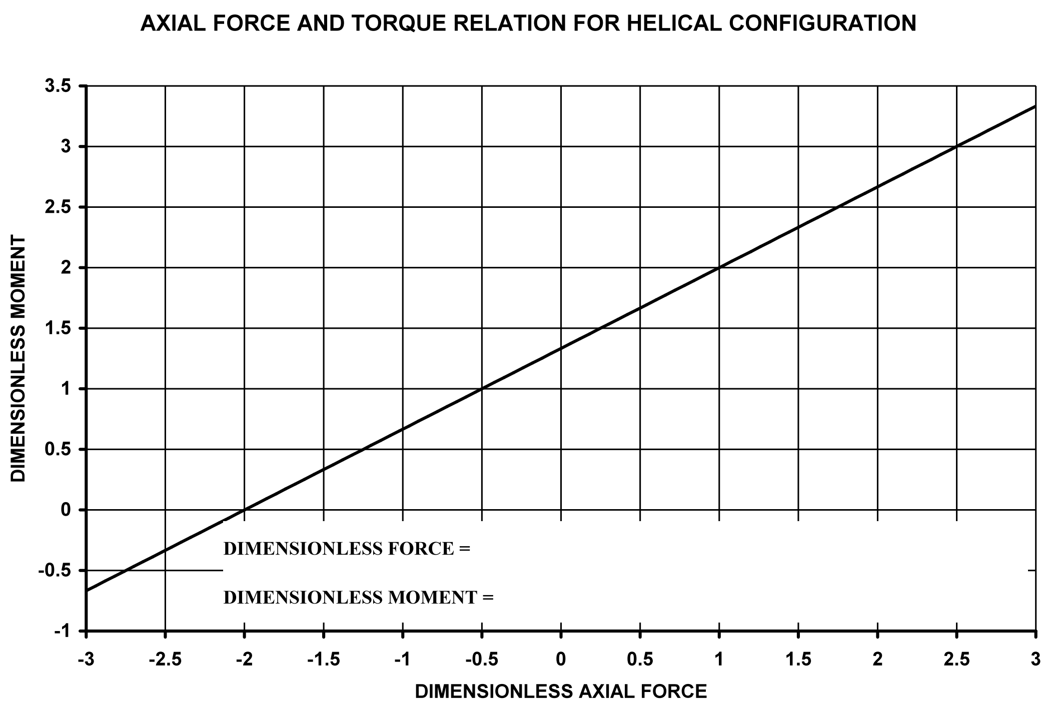
$$\frac{M_{LATERAL} \cdot R}{EI} = \left(\frac{M_\tau \cdot R}{EI} + \sin^2\alpha \cdot \frac{F_\eta \cdot R^2}{EI \cdot \sin^2\alpha}\right) \cdot \sin\alpha + \left(\frac{M_\eta \cdot R}{EI} - \sin^2\alpha \cdot \frac{F_\tau \cdot R^2}{EI \cdot \sin^2\alpha}\right) \cdot \cos\alpha$$

FURTHER CONSIDERATIONS

If the condition,

$$\frac{d}{d\alpha}\left(\frac{R^3 \cdot w}{EI}\right)\Bigg|_{\alpha<<1} = 0$$

is adopted to resolve the value of $\frac{R^3 \cdot w}{EI}$. Then the following curve results. This curve is essentially independent of the value of α up to 5 degrees



When this condition is introduced, M_τ set to zero and the solution restricted to values of $\alpha << 1$, the solution becomes,

$$\frac{F_\tau \cdot R^2}{EI \cdot \sin^2 \alpha} = -2$$

$$\frac{w \cdot R^3}{EI \cdot \sin^4 \alpha} = +1$$

$$\frac{F_\eta \cdot R^2}{EI \cdot \sin^2 \alpha} = +\sin \alpha$$

$$\frac{F_\eta}{F_\tau} = -\tfrac{1}{2} \cdot \sin \alpha$$

$$\frac{M_\eta \cdot R}{EI} = -\sin^2\alpha$$

$$\frac{F_{AXIAL} \cdot R^2}{EI \cdot \sin^2\alpha} = -2$$

$$\frac{M_{AXIAL} \cdot R}{EI} = 0$$

$$\frac{F_{LATERAL} \cdot R^2}{EI \cdot \sin^2\alpha} = -\sin\alpha$$

$$\frac{M_{LATERAL} \cdot R}{EI} = +\sin^2\alpha$$

ADDENDUM

On page 3 above, the term, $\frac{d\varphi}{ds} - \frac{d\vartheta}{ds}$, is introduced as the elastic twist angle per unit length. When

$$R \cdot \frac{d\vartheta}{ds} = \sin\alpha$$

is used to eliminate $\frac{d\vartheta}{ds}$, the expression becomes $\frac{d\varphi}{ds} - \frac{\sin\alpha}{R}$. The derivation of this expression for the twist per unit length can be found in Love's treatise*. A physical description of the derivation is that two types of "rotation" are defined so that their difference gives the desired twist per unit length. The first rotation, ϑ, relies, in the current problem, on bending the axis of an initially straight rod into a helix so that every line in the rod that was initially perpendicular to the cross section remains perpendicular to the cross section. A second rotation, φ, is defined as the cumulative rotation of the actual cross section while moving along the deformed axis. Clearly, if $\varphi = \vartheta$ there is no component of elastic moment directed along the deformed rod axis. This explains why the measure of twist is based on the difference between these angles.

* A. E. H. Love, *A Treatise on the Mathematical Theory of Elasticity*, Fourth Edition, Dover Publications, New York, 1944, Sections 252 & 253

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