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SOME PROPERTIES OF ELLIPSES

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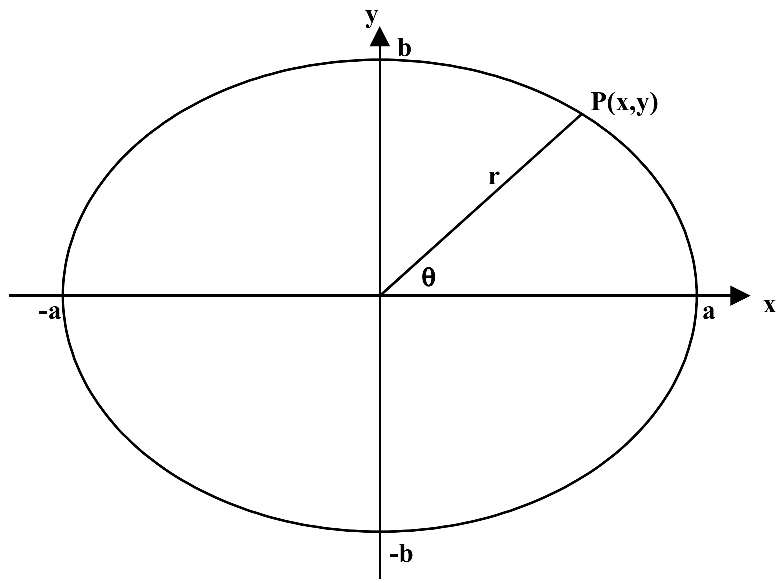

An ellipse is defined in the usual way by,

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1 \qquad 1$$

where a and b are the semi-axes. The following results are obvious and often useful,

$$y = b \cdot \sqrt{1 - \left(\frac{x}{a}\right)^2} \qquad 2$$

$$r = \sqrt{x^2 + y^2} = b \cdot \sqrt{1 + x^2 \cdot \left(\frac{1}{b^2} - \frac{1}{a^2}\right)} \qquad 3$$

$$\theta = \arctan\left(\frac{y}{x}\right) = \arctan\left(b \cdot \sqrt{\frac{1}{x^2} - \frac{1}{a^2}}\right) \qquad 4$$

$$x = \frac{1}{\sqrt{\frac{\tan^2 \theta}{b^2} + \frac{1}{a^2}}} \qquad 5$$

The area of the ellipse, A, is given by,

$$A = \pi \cdot a \cdot b \tag{6}$$

The area moment of inertia, I_{xx} , about the x-axis is given by,

$$I_{xx} = \tfrac{1}{4} \cdot \pi \cdot b^3 \cdot a \tag{7}$$

When the middle surface of a thin-walled tube of wall thickness t has a cross section that is elliptical, the area moment of inertia about the x-axis is approximated by,

$$I_{xx} \cong \tfrac{1}{4} \cdot \pi \cdot b^2 \cdot t \cdot (b + 3 \cdot a) \tag{8}$$

The perimeter, P, of the ellipse is given by,

$$P = 4 \cdot \int_{x=0}^{a} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \cdot dx$$

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After some manipulations, Equation 9 becomes,

$$P = 4 \cdot \int_{x=0}^{a} \frac{\sqrt{1 + \left[\left(\frac{b}{a}\right)^2 - 1\right] \cdot \left(\frac{x}{a}\right)^2}}{\sqrt{1 - \left(\frac{x}{a}\right)^2}} \cdot dx \tag{10}$$

Let

$$\xi = \frac{x}{a} \qquad \text{and} \qquad m = 1 - \left(\frac{b}{a}\right)^2$$

and assume $b<a$ so that,

$$P = 4 \cdot a \cdot \int_{\xi=0}^{1} \sqrt{\frac{1 - m \cdot \xi^2}{1 - \xi^2}} \cdot d\xi \equiv 4 \cdot a \cdot E(m) \tag{11}$$

where E(m) is the complete elliptical integral of the second kind. The function $a \cdot E(m)$ has the physical interpretation of being the perimeter length in the first quadrant between the points (a,0) and (0,b). This integral is tabulated in many places. Elementary geometric considerations show that,

$$E(0) = \frac{\pi}{2} \qquad (m = 0 \text{ for a circle})$$

$$E(1) = 1 \qquad (m = 1 \text{ for a line on the x-axis})$$

The figure below shows that E(m) is nearly a straight line for m in the range –0.2 < m < 0.2 . Two very useful approximations to this function are given by,

$$E(m) \cong 1. + 0.4630151\cdot(1. - m) + 0.1077812\cdot(1. - m)^2$$
$$+ \left[0.2452727\cdot(1. - m) + 0.0412496\cdot(1. - m)^2\right]\cdot \ln\left(\frac{1}{(1. - m)}\right) \qquad (0 < m < 1) \tag{12}$$

$$E(m) \cong 1. + 0.44325141463\cdot(1. - m) + 0.0626060122\cdot(1. - m)^2$$
$$+ 0.04757383546\cdot(1. - m)^3 + 0.01736506451\cdot(1. - m)^4$$
$$+ \left[\begin{array}{l} 0.2499836831\cdot(1. - m) + 0.09200180037\cdot(1. - m)^2 \\ + 0.04069697526\cdot(1. - m)^3 + 0.00526449639\cdot(1. - m)^4 \end{array}\right]\cdot \ln\left(\frac{1}{(1. - m)}\right) \quad (0<m<1) \tag{13}$$

The absolute value of the error in the first expression is less than 4E-5 while the corresponding error from the second expression is less than 2E-8.

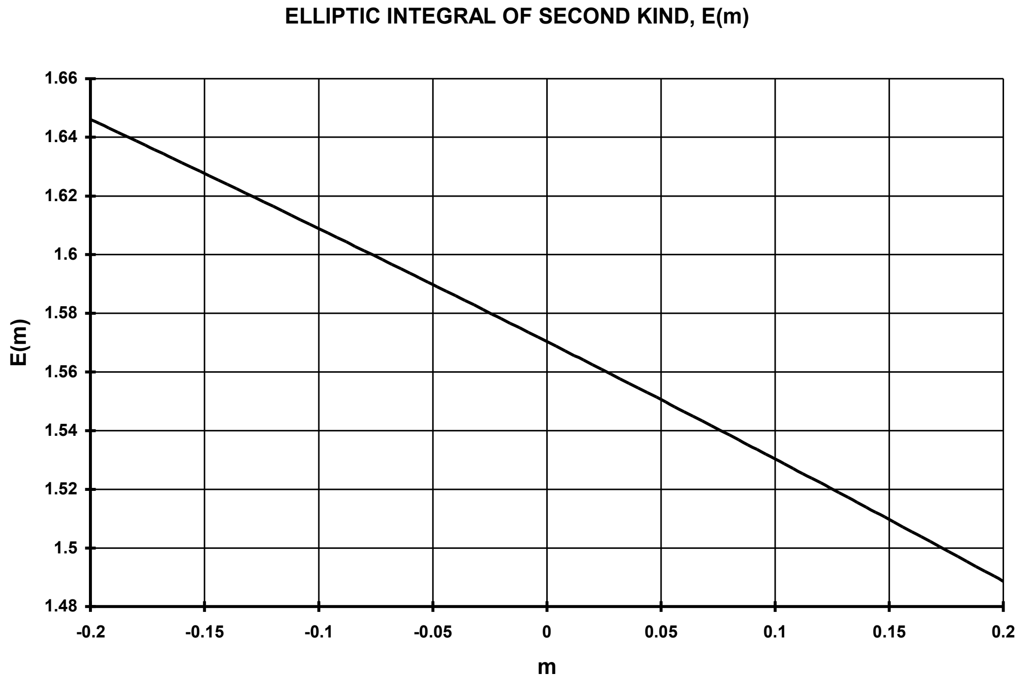
In *Numerical Recipes* by William H. Press et al, E(m) is written in the form,

$$E(m) = \int_0^{\frac{\pi}{2}} \sqrt{1 - k^2 \cdot \sin^2\theta} \cdot d\theta \qquad k^2 = m \tag{14}$$

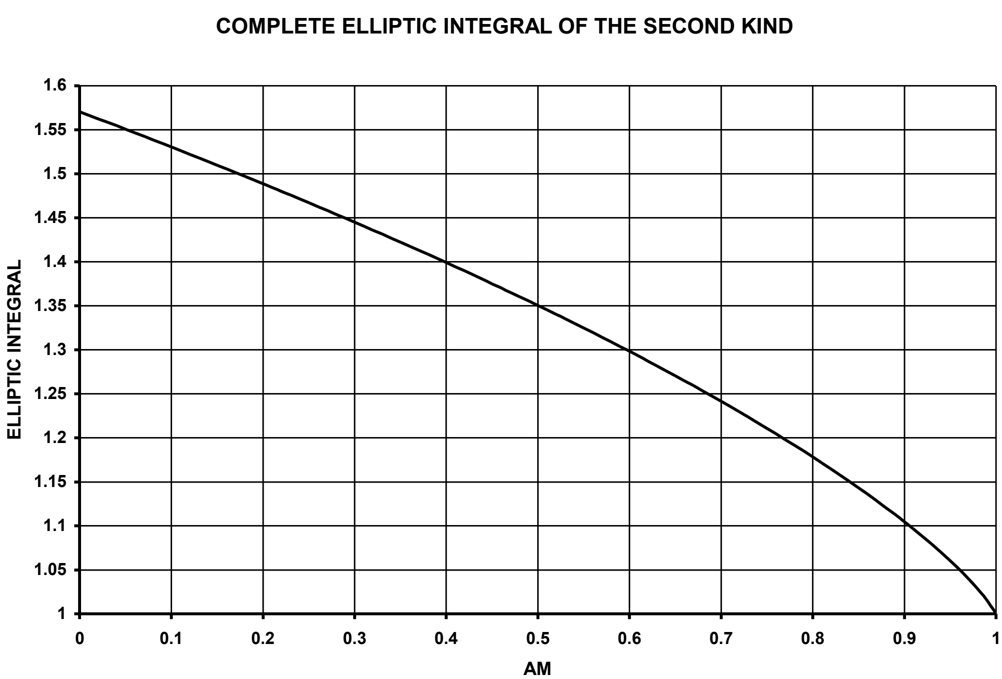
and this book gives a computer code for a rather general function that includes E(m) as,

$$E(m) = \mathrm{cel}(\sqrt{1 - m}, 1., 1., 1 - m) \tag{15}$$

see page 185.



The following figure shows the elliptic integral for the range of arguments from 0 to 1.



In the sketch above the point P(x,y) is shown as a point on the ellipse in the first quadrant. The distance along the ellipse in the first quadrant from (a,0) to P(x,y) is denoted

$a \cdot \overline{E}(\theta, m)$. The function, $\overline{E}(\theta, m)$, is called the Legendre elliptic integral of the second kind and it is defined by any of the following expressions,

$$\overline{E}(\theta, m) = \int_0^{\theta} \sqrt{1 - k^2 \cdot \sin^2\theta} \cdot d\theta \qquad k^2 = m$$

$$= \int_0^{\sin\theta} \sqrt{\frac{1 - m \cdot \xi^2}{1 - \xi^2}} \cdot d\xi \qquad 16$$

$$= \int_0^{\tan\theta} \frac{\sqrt{1 + k_c{}^2 \cdot \eta^2}}{\left(1 + \eta^2\right)^{\frac{3}{2}}} \cdot d\eta \qquad k_c{}^2 = 1 - k^2 = 1 - m$$

Numerical Recipes also has this function in the form of a computer code so that,

$$\overline{E}(\theta, m) = el2(\tan\theta, \sqrt{1 - m}, 1., 1 - m)$$
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Clearly,

$$E(m) = \overline{E}\left(\frac{\pi}{2}, m\right) \qquad 18$$

Consider a circle of radius RO. The perimeter of this circle is $2 \cdot \pi \cdot RO$. Now consider an ellipse with semi-axes defined by,

$$a = RO + \Delta R \qquad 19$$

$$b = RO - \Delta R \qquad 20$$

The ovality of a tubular is defined by $\frac{R_{max} - R_{min}}{R_{max} + R_{min}} = \frac{\Delta R}{RO}$ and is designated as OV. The expression for m, in terms of OV, becomes,

$$m = 1 - \left(\frac{1 - OV}{1 + OV}\right)^2 = \frac{4 \cdot OV}{(1 + OV)^2} \qquad 21$$

and,

$$a = RO \cdot (1 + OV) \qquad 22$$

$$b = RO \cdot (1 - OV) \qquad 23$$

Recalling that E(m) is nearly linear with m in the neighborhood of m = 0, an approximation for E(m) is,

$$E(m) \cong \frac{\pi}{2} \cdot \left[1 - \frac{OV}{(1+OV)^2}\right] \qquad 24$$

The area of the ellipse is given by,

A = area of the ellipse = $\pi \cdot RO^2 \cdot (1 - OV^2)$ 25

while

$$P = \text{perimeter of the ellipse} = 4 \cdot RO \cdot (1+OV) \cdot E\left[1 - \left(\frac{1-OV}{1+OV}\right)^2\right]$$

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$$\frac{x}{RO} = \frac{1}{\sqrt{\left(\frac{1}{1+OV}\right)^2 + \left(\frac{\tan\theta}{1-OV}\right)^2}} \qquad 27$$

The perimeter of the ellipse is greater than the perimeter of the circle of radius RO. The "mean hoop strain", $\varepsilon_{hoop,}$ associated changing from the circle to the ellipse is given by,

$$\varepsilon_{hoop} = \frac{P - 2 \cdot \pi \cdot RO}{2 \cdot \pi \cdot RO} = \frac{2}{\pi} \cdot (1+OV) \cdot E\left[1 - \left(\frac{1-OV}{1+OV}\right)^2\right] - 1 \qquad 28$$

When the perimeter is not permitted to change length while changing from a circle to an ellipse, the above strain must be removed. This may be accomplished by "shrinking" the cross section uniformly. Each dimension is divided by $(1 + e_{hoop})$. When the shrinking is done the area, A, of the ellipse is,

$A = \pi \cdot RO^2 \cdot (1 - OV^2) / (1 + \varepsilon_{hoop})^2$ 29

In many applications other characteristics of an ellipse are required. The following derivation is to determine the curvature as a function of x in the first quadrant for the ellipse defined by $a = RO + \Delta R$ and $b = RO - \Delta R$ so that,

$$y = (1-OV) \cdot RO \cdot \sqrt{1 - \frac{x^2}{(1+OV)^2 \cdot RO^2}} \qquad 30$$

and then,

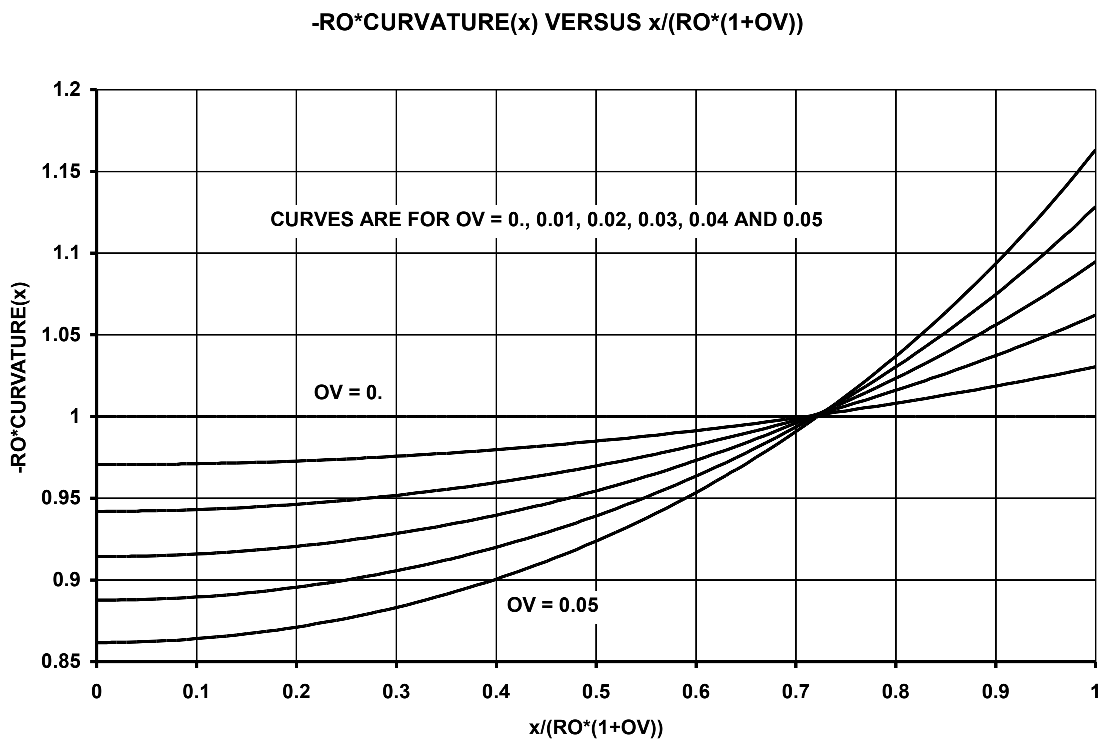
$$y' = -\frac{1-OV}{1+OV} \cdot \frac{x}{\sqrt{(1+OV)^2 \cdot RO^2 - x^2}} \qquad 31$$

$$y'' = -\frac{1-OV}{1+OV}\cdot\frac{RO^2\cdot(1+OV)^2}{\left[(1+OV)^2\cdot RO^2 - x^2\right]^{\frac{3}{2}}} = \frac{RO^2\cdot\left(1-OV^2\right)}{\left[(1+OV)^2\cdot RO^2 - x^2\right]^{\frac{3}{2}}} \qquad 32$$

Denote κ(x) as the curvature so that,

$$\kappa(x) = \frac{y''}{\left(1+(y')^2\right)^{\frac{3}{2}}} = -\frac{1-OV}{1+OV}\cdot\frac{RO^2\cdot(1+OV)^2}{\left[(1+OV)^2\cdot RO^2 - x^2 + \frac{(1-OV)^2}{(1+OV)^2}\cdot x^2\right]^{\frac{3}{2}}} \qquad 33$$

The curve below is based on Equation 33,



The following expressions for curvature at the semi-axes are sometimes useful,

$$\kappa(0) = -\frac{1-OV}{1+OV}\cdot\frac{1}{RO\cdot(1+OV)} \qquad 34$$

$$\kappa((1+OV)\cdot RO) = -\frac{1+OV}{1-OV}\cdot\frac{1}{RO\cdot(1-OV)}$$

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or,

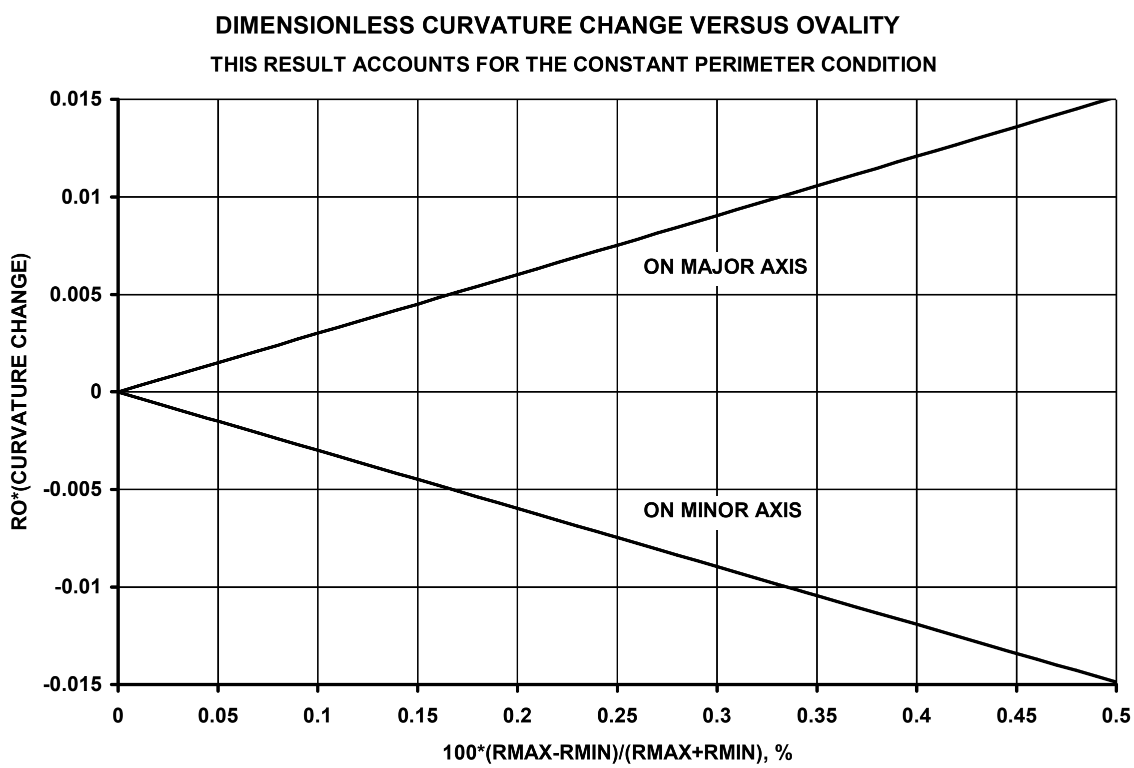
$$\kappa(x) = -\frac{a \cdot b}{\left(a^2 - \left(a^2 - b^2\right)\cdot\left(\frac{x}{a}\right)^2\right)^{\frac{3}{2}}} \qquad 33a$$

$$\kappa(0) = -\frac{b}{a^2} \qquad 34a$$

$$\kappa(a) = -\frac{a}{b^2}$$
35a

If the "shrinkage" mentioned above is applied to the ellipse then the curvatures should be multiplied by $(1 + \varepsilon_{hoop})$. The curves below show the changes of dimensionless curvature with ovality, taking into account the shrinkage. These curves show that, for small ovality changes from a circular section, the curvature changes can be approximated using,

$$RO \cdot (\text{curvature change}) \cong \pm 3 \cdot OV \qquad 36$$

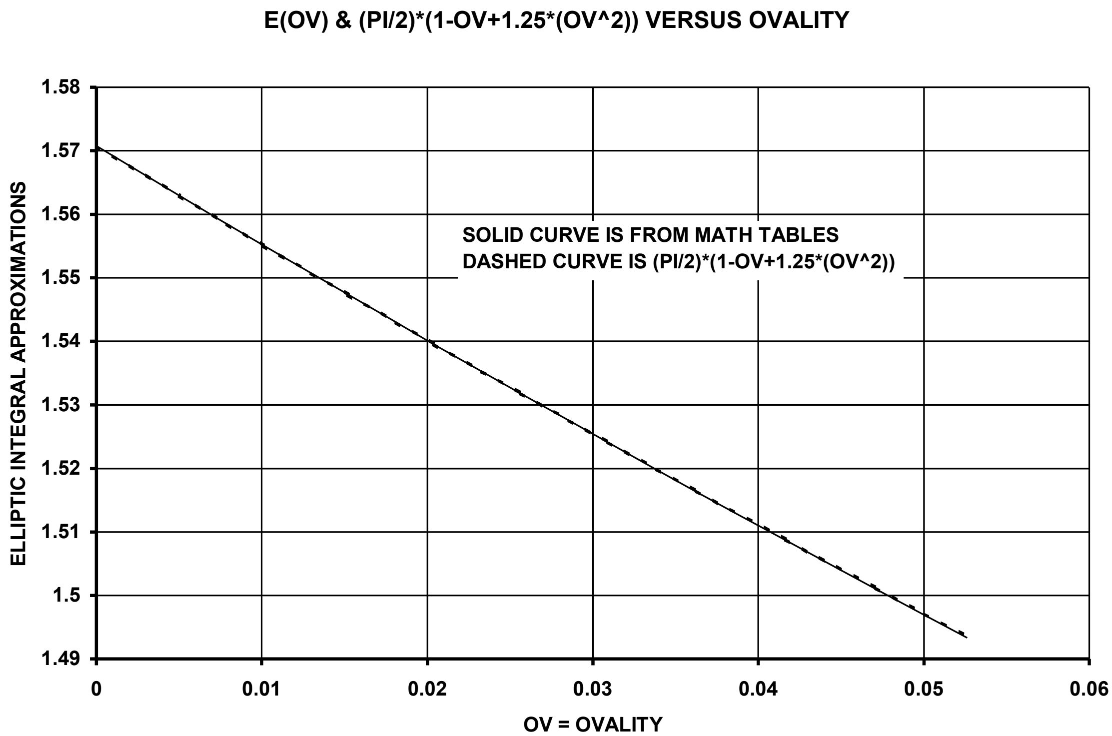


A useful formulation for finding the change in area from a circle to an ellipse of the same perimeter with ovality OV is given below using the approximation,

E(OV) = complete elliptical integral of the second kind as a function of ovality

$$= \mathbf{E}\left[1 - \left(\frac{1 - \mathrm{OV}}{1 + \mathrm{OV}}\right)^2\right] \cong \frac{\pi}{2} \cdot \left(1 - \mathrm{OV} + \frac{5}{4} \cdot \mathrm{OV}^2\right) \qquad 37$$

The curves below indicate the quality of the match with the calculated values using *Numerical Recipes* .



Consider the circle of radius rO. Maintain the perimeter constant and distort the circle to an ellipse of ovality OV. Denote the major and minor semi-axes as,

$$a = \mathrm{rO} + \Delta \mathrm{r1} + \Delta \mathrm{r2} \qquad 38$$

$$b = \mathrm{rO} + \Delta \mathrm{r1} - \Delta \mathrm{r2} \qquad 39$$

The original ovality is zero and the final ovality is,

$$\mathrm{OV} = \frac{\Delta \mathrm{r2}}{\mathrm{rO} + \Delta \mathrm{r1}} \qquad 40$$

The original area is $\pi \cdot \mathrm{rO}^2$ and the final area, AF, is,

$$\mathrm{AF} = \pi \cdot \mathrm{rO}^2 \cdot \left(1 + \frac{\Delta \mathrm{r1}}{\mathrm{rO}}\right)^2 \cdot \left(1 - \mathrm{OV}^2\right) \qquad 41$$

The original perimeter is $2 \cdot \pi \cdot rO$ and the final perimeter, PF, is,

$$PF = 4 \cdot a \cdot \mathbf{E}(OV) \cong 2 \cdot \pi \cdot rO \cdot \left(1 + \frac{\Delta r1}{rO}\right) \cdot (1 + OV) \cdot \left(1 - OV + \frac{5}{4} \cdot OV^2\right) \qquad 42$$

The condition that the perimeter is unchanged gives,

$$1 + \frac{\Delta r1}{rO} \cong \frac{1}{\left(1 - OV + \frac{5}{4} \cdot OV^2\right) \cdot (1 + OV)} \cong 1 - \frac{1}{4} \cdot OV^2 \qquad 43$$

Substituting Equation 42 into Equation 40 to eliminate $\Delta r1$ gives,

$$AF \cong \pi \cdot rO^2 \cdot \left(1 - \frac{1}{4} \cdot OV^2\right)^2 \cdot \left(1 - OV^2\right) \cong \pi \cdot rO^2 \cdot \left(1 - \frac{3}{2} \cdot OV^2\right) \qquad 44$$

Therefore,

$$\frac{AF - \pi \cdot rO^2}{\pi \cdot rO^2} = \frac{(\text{change of area})}{(\text{original area})} \cong -\frac{3}{2} \cdot OV^2 \qquad 45$$

In the literature concerning "Plates and Shells", the mathematical modeling for ovalling of a cylinder is usually different from the elliptical shape considered above. The common form for ovalling is that the radial displacement from the initially circular cross section is proportional to cos θ. This formulation is presented below and then compared with the results presented above for ellipses.

Consider a circle of radius rO with points A and B fixed on the perimeter. The points separated by the differential polar angle $d\phi$. Now distort the circle by specifying displacements in the plane of the circle. The displacements are given as an outward radial displacement, w_r, and a tangential displacement, w_t. Each of these displacements is a function of ϕ. Note that, at each fixed point, the radial and tangential directions are referred to the initial, circular configuration. The initial distance, ds_i, between points A and B is given by,

$$ds_i = rO \cdot d\phi \qquad 46$$

The distance, ds_f, between the points after the circle is distorted may be related to ds_i using straightforward analysis (given at the end of this appendix) with the result that the hoop strain, $\varepsilon_{hoop,}$ is given by,

$$\varepsilon_{hoop} = \frac{ds_f}{ds_i} - 1 = \frac{w_r + w_t'}{rO} - \frac{1}{2} \cdot \left(\frac{w_r + w_t'}{rO}\right)^2 + \frac{w_t^2 + w_r^2 - 2 \cdot w_t \cdot w_r' + 2 \cdot w_r \cdot w_t' + w_r'^2 + w_t'^2}{2 \cdot rO^2} \qquad 47$$

where a prime signifies differentiation with respect to ϕ. When the inextensional theory of cylindrical shells is studied, the terms in ε_{hoop} that are linear in w_r and w_t are set to zero so that,

$$w_r = - w_t' \qquad 48$$

and then the hoop strain is given by,

$$\varepsilon_{hoop} = \frac{(w_t + w_t'')^2}{2 \cdot rO^2} \qquad 49$$

This strain results because the usual inextensional theory considers only the linear displacement terms in the strain equations.

If w_t is taken as,

$$w_t = c \cdot \sin(2 \cdot \phi) \qquad 50$$

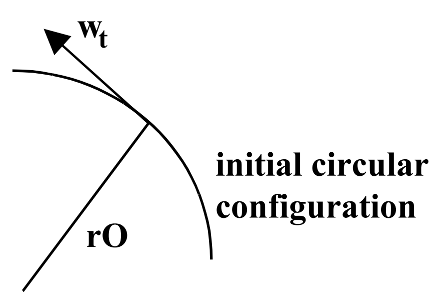
so that, from Equation 48,

$$w_r = -2 \cdot c \cdot \cos(2 \cdot \phi) \qquad 51$$

and ε_{hoop} is given by,

$$\varepsilon_{hoop} = \frac{c^2}{2 \cdot rO^2} \cdot 9 \cdot \sin^2(2 \cdot \phi) \qquad 52$$

Note that this strain includes the term $\frac{w_t^2}{2 \cdot rO^2}$ in Equation 47. This term shows that when $w_r = 0$ and w_t is constant (this satisfies Equation 48), there is a non-trivial hoop strain. This result is correct and follows from the definition of w_t being in the tangential direction for the circular configuration, see sketch below,



The hoop strain induced by this displacement is $\dfrac{\sqrt{rO^2 + w_t^{\,2}} - rO}{rO}$. For w_t << rO, this hoop strain is $\dfrac{w_t^{\,2}}{2 \cdot rO^2}$ and equals the strain in question in Equation 47. The perimeter change ΔP from the strain of Equation 52 is,

$$\Delta P = \int_{\phi = 0}^{\phi = 2\cdot\pi} \varepsilon_{hoop} \cdot rO \cdot d\phi = \frac{9 \cdot \pi \cdot c^2}{2 \cdot rO} \qquad 53$$

If the perimeter is reduced by reducing the radius by Δr to offset the perimeter growth ΔP, the result is,

$$\Delta r = -\frac{\Delta P}{2 \cdot \pi} = -\frac{9 \cdot c^2}{4 \cdot rO} \qquad 54$$

so that the area change, ΔA_P, for this perimeter adjustment is approximated by,

$$\Delta A_P \cong 2 \cdot \pi \cdot rO \cdot \Delta r = -\frac{9 \cdot \pi \cdot c^2}{2} \qquad 55$$

The changes of areas, ΔA_r and ΔA_t , associated with w_r and w_t , respectively are, to second order terms in w_r and w_t,

$$\Delta A_r = \int_{\phi = 0}^{\phi = 2\cdot\pi} \tfrac{1}{2} \cdot (rO + w_r)^2 \cdot \left(1 + \frac{w_t'}{rO}\right) \cdot d\phi - \pi \cdot rO^2 = -2 \cdot \pi \cdot c^2 \qquad 56$$

$$\Delta A_t = 2 \cdot \pi \cdot rO \int_{\phi = 0}^{\phi = 2\cdot\pi} \tfrac{1}{2} \cdot \frac{w_t^{\,2}}{rO^2} \cdot \frac{rO \cdot d\phi}{2 \cdot \pi} = +\frac{\pi}{2} \cdot c^2 \qquad 57$$

Consequently, when the perimeter is restricted to being inextensible through second order terms in w_r and w_t, the net change in area, ΔArea, is approximated by,

$$\Delta Area \cong \Delta A_P + \Delta A_r + \Delta A_t = -\frac{9 \cdot \pi \cdot c^2}{2} - 2 \cdot \pi \cdot c^2 + \frac{\pi \cdot c^2}{2} = -6 \cdot \pi \cdot c^2 \qquad 59$$

or

$$\frac{\Delta\, Area}{\pi \cdot rO^2} = -6 \cdot \left(\frac{c}{rO}\right)^2 \qquad 60$$

Now, the “ovality” for the distorted configuration is given as “OV” where,

$$"OV" = \frac{(\text{maximum diameter}) - (\text{minimum diameter})}{(\text{maximum diameter}) + (\text{minimum diameter})} = \frac{2 \cdot c}{rO} \qquad 61$$

Combining Equations 60 and 61 yields,

$$\frac{\Delta\, Area}{\pi \cdot rO^2} = -\frac{3}{2} \cdot "OV"^2 \qquad 60a$$

This result agrees with Equation 45 for the case of an ellipse. Thus, the sinusoidal formulation is, to second order terms in w_r and w_t, equivalent to the elliptical formulation. At the practical level, the results show that classical collapse analysis of thin-walled cylindrical shells may be simplified by using the sinusoidal formulation to find the changes in curvature while using the elliptical formulation to find the change in area.

It is interesting to note that when Equation 50 is replaced by,

$$w_t = c \cdot \sin(n \cdot \phi) \qquad 61a$$

then,

$$w_r = -n \cdot c \cdot \cos(n \cdot \phi) \qquad 62$$

$$\varepsilon_{hoop} = \frac{c^2}{2 \cdot rO^2} \cdot \left(n^2 - 1\right)^2 \cdot \sin^2(n \cdot \phi) \qquad 63$$

$$\Delta P = \frac{\left(n^2 - 1\right)^2 \cdot \pi \cdot c^2}{2 \cdot rO} \qquad 64$$

$$\Delta r = -\frac{\left(n^2 - 1\right)^2 \cdot c^2}{4 \cdot rO} \qquad 65$$

$$\Delta A_P = -\frac{\left(n^2 - 1\right)^2 \cdot \pi \cdot c^2}{2} \qquad 66$$

$$\Delta A_r = -\frac{n^2}{2} \cdot \pi \cdot c^2 \qquad 67$$

$$\Delta A_t = +\frac{\pi}{2} \cdot c^2 \qquad 68$$

$$\Delta Area \cong -\frac{n^2 \cdot \left(n^2 - 1\right)}{2} \cdot \pi \cdot c^2 \qquad 69$$

$$\frac{\Delta \text{Area}}{\pi \cdot rO^2} = -\frac{n^2 \cdot \left(n^2 - 1\right)}{2} \cdot \left(\frac{c}{rO}\right)^2 \qquad 70$$

$$"\overline{OV}" = \frac{(\text{maximum radius}) - (\text{minimum radius})}{(\text{maximum radius}) + (\text{minimum radius})} = \frac{n \cdot c}{rO} \qquad 71$$

$$\frac{\Delta \text{Area}}{\pi \cdot rO^2} = -\frac{n^2 - 1}{2} \cdot "\overline{OV}"^2 \qquad 72$$

As a concluding observation the derivation of the hoop strain, ε_{hoop}, in terms of the radial and tangential displacements, $w_r(\phi)$ and $w_t(\phi)$, is presented. First consider the points A and B defined on page 10 in the undeformed configuration. Using the conventional Cartesian coordinate system, the x and y coordinates if the two points are given by,

$$\begin{aligned} x_A &= rO \cdot \cos\phi \\ y_A &= rO \cdot \sin\phi \end{aligned} \qquad 73$$

and

$$\begin{aligned} x_A &= rO \cdot \cos(\phi + d\phi) \\ y_A &= rO \cdot \sin(\phi + d\phi) \end{aligned} \qquad 74$$

The differential distance between points A and B in the undeformed configuration, ds_i, is obtained using the identities,

$$\begin{aligned} \sin(a+b) &= \sin a \cdot \cos b + \cos a \cdot \sin b \\ \cos(a+b) &= \cos a \cdot \cos b - \sin a \cdot \sin b \end{aligned} \qquad 75$$

so that,

$$\begin{aligned} (ds_i)^2 &= rO^2 \cdot (\cos(\phi + d\phi) - \cos\phi)^2 + rO^2 \cdot (\sin(\phi + d\phi) - \sin\phi)^2 \\ &= rO^2 \cdot \left(\cos^2\phi + (-\sin\phi)^2\right) \cdot d\phi^2 = rO^2 \cdot d\phi^2 \end{aligned} \qquad 76$$

In the deformed configuration the positions of material fixed points A and B are changed to the positions $\widetilde{A}$ and $\widetilde{B}$ with the radial and tangential displacements, w_r and w_t. These displacements and their derivatives with respect to ϕ are assumed to be very small compared to rO. The new positions are now,

$$x_{\tilde{A}} = (rO + w_r) \cdot \cos\phi - w_t \cdot \sin\phi$$
$$y_{\tilde{A}} = (rO + w_r) \cdot \sin\phi + w_t \cdot \cos\phi \qquad 77$$

and,

$$x_{\tilde{B}} = (rO + w_r + w_r' \cdot d\phi) \cdot \cos(\phi + d\phi) - (w_t + w_t' \cdot d\phi) \cdot \sin(\phi + d\phi)$$
$$y_{\tilde{B}} = (rO + w_r + w_r' \cdot d\phi) \cdot \sin(\phi + d\phi) + (w_t + w_t' \cdot d\phi) \cdot \cos(\phi + d\phi) \qquad 78$$

so that, through first order terms in dϕ,

$$x_{\tilde{B}} - x_{\tilde{A}} = [(w_r' - w_t) \cdot \cos\phi - (rO + w_r + w_t') \cdot \sin\phi] \cdot d\phi$$
$$y_{\tilde{B}} - y_{\tilde{A}} = [(w_r' - w_t) \cdot \sin\phi + (rO + w_r + w_t') \cdot \cos\phi] \cdot d\phi \qquad 79$$

and then,

$$(ds_f)^2 = (x_{\tilde{B}} - x_{\tilde{A}})^2 + (y_{\tilde{B}} - y_{\tilde{A}})^2$$
$$= \left[rO^2 + 2 \cdot rO \cdot (w_r + w_t') + w_r'^2 - 2 \cdot w_r' \cdot w_t + w_t^2 + w_r^2 + 2 \cdot w_r \cdot w_t' + w_t'^2\right] \cdot (d\phi)^2 \qquad 80$$

Recalling that the displacements and their derivatives are small compared to rO, the hoop strain may be approximated as follows,

$$\varepsilon_{hoop} = \sqrt{\frac{(ds_f)^2}{rO^2 \cdot (d\phi)^2}} - 1$$
$$= \frac{w_r + w_t'}{rO} - \frac{1}{2} \cdot \left(\frac{w_r + w_t'}{rO}\right)^2 + \frac{w_r^2 + w_r'^2 + 2 \cdot w_r \cdot w_t' - 2 \cdot w_r' \cdot w_t + w_t^2 + w_t'^2}{2 \cdot rO^2} \qquad 81$$

Equation 81 is the same as Equation 47

AN ILLUSTRATIVE PROBLEM USING EQUATION 59 ABOVE: BUCKLING OF A THIN CIRCULAR RING, SUBJECTED TO A UNIFORM RADIAL LOADING

R	= mean radius of ring
w	= radial component of displacement
EI	= bending stiffness for ring
ΔP	= radial load per unit length
M	= bending moment
θ	= angular coordinate
U	= strain energy
V	= potential energy of load
PE	= total potential energy
δPE	= first variation of PE
$\delta^2 PE$	= second variation of PE

$$M = \frac{EI}{R^2} \cdot \left(\frac{d^2 w}{dx^2} + w \right)$$

$$U = \int_{\theta=0}^{2\cdot\pi} \frac{M^2 \cdot R}{2 \cdot EI} \cdot d\theta = \frac{EI}{2 \cdot R^3} \cdot \int_{\theta=0}^{2\cdot\pi} \left(\frac{d^2 w}{dx^2} + w \right)^2 \cdot d\theta$$

$$w = w_O \cdot \cos(2 \cdot \theta)$$

$$U = \frac{9 \cdot \pi}{2} \cdot \frac{EI \cdot w_O^{\,2}}{R^3}$$

$$V = -\Delta P \cdot 6 \cdot \pi \cdot \left(\tfrac{1}{4} \cdot w_O^{\,2} \right) = -\tfrac{3}{2} \cdot \pi \cdot \Delta P \cdot w_O^{\,2} \qquad \text{from Equation 59 above}$$

$$PE = U + V = \left(\frac{9 \cdot \pi}{2} \cdot \frac{EI}{R^3} - \frac{3 \cdot \pi}{2} \cdot \Delta P \right) \cdot w_O^{\,2}$$

Note there are no linear terms for w_O in the above equation. This is expected when a buckling load is being predicted. The equilibrium solution is obtained by setting δPE to zero so that,

$$\delta PE = \left(\frac{9 \cdot \pi}{2} \cdot \frac{EI}{R^3} - \frac{3 \cdot \pi}{2} \cdot \Delta P \right) \cdot w_O \cdot \delta w_O = 0$$

and,

$$\Delta P = \frac{3 \cdot EI}{R^3}$$

The physical interpretation of the last equation is that for the assumed displacement and load distribution the only equilibrium solution is given by this equation. The stability of this solution may be found by evaluating $\delta^2 PE$ and,

$$\delta^2 PE = \frac{3 \cdot \pi}{2} \cdot \left(\frac{3 \cdot EI}{R^3} - \Delta P \right) \cdot \cdot \left(\delta w_O \right)^2$$

The solution is stable whenever $\delta^2 PE > 0$. Consequently,

Limit of stability = buckling load = $\Delta P = \dfrac{3 \cdot EI}{R^3}$

This solution agrees with the solution in Stephen P. Timoshenko and James M. Gere's *Theory of Elastic Stability*, Second Edition, 1961, Section 7.4. The advantage of using Equation 59 for the area change is that the perimeter of the ring is unaltered *through second orders terms in* w_O and the extensional contribution to U can be ignored in the prediction of the usual buckling pressure differential.

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