BLADE ENERGY PARTNERS PLASTICITY LECTURE NOTES

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PRELIMINARY REMARKS

These are lecture notes prepared for the Blade Energy Partners Film "Theoretical and Applied Plasticity". Several sources were used to compile the notes so the format varies from section to section. These notes are not self-contained so considerable explanatory material would be required to fill in much of what is presented below. They should be regarded as a supplement to the film for persons wanting a bit more detail than the film provides.

BLADE ENERGY PARTNERS

THEORETICAL AND APPLIED PLASTICITY COURSE, MAY 9, 2011

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INTRODUCTION

Principal Texts,

1. William Prager and Philip G. Hodge, Jr., *Theory of Perfectly Plastic Solids*, Dover Publications, 1968 (originally published by John Wiley & Sons, Inc., 1951)
2. Rodney Hill, *The Mathematical Theory of Plasticity*, Oxford at the Clarendon Press, 1950
3. Ted L. Anderson, *Fracture Mechanics*, Third Edition, Taylor & Francis, 2005
4. Philip G. Hodge, Jr., *Plastic Analysis of Structures*, McGraw-Hill Book Company, 1959

Plastic Instability of a Tension Bar

The derivation presented here is taken from Hill, Reference 2. This derivation considers a cylindrical bar subjected to tension parallel to its axis. For ductile, metallic bars it is well known that, for a monotonically increasing tension load which started at the unloaded condition, the load reaches a maximum and then decreases before the bar is separated into two pieces. The analysis given here relates the occurrence of the maximum load to a property of the true stress versus engineering strain curve for the material being tested.

While the bar is being loaded let,

σ = current true stress = (axial load)/(current cross-sectional area)
l = current length of the bar
l_o = original length of the unloaded bar
$\bar{\varepsilon}$ = currenttruestrain=$\ln(l/l_o)$
$\bar{e}$ = current engineering strain = $l/l_o - 1$
note that $\bar{\varepsilon} = \ln(1 + \bar{e})$ and $\bar{e} = e^{\bar{\varepsilon}} - 1$
A = current cross-sectional area
A_o = original cross-sectional area of the unloaded bar
L = axial load (positive when tensile)

The volume of the bar is $A \cdot l$. Assume the elastic strains are negligible compared to the plastic strains when the maximum load is reached. Further, make the usual assumption that plastic deformation does not alter the material volume. Then

$$d(A \cdot l) = 0 \rightarrow l \cdot dA + A \cdot dl = 0 \qquad \text{AI.1}$$

The condition that the axial load reaches a maximum value implies that

$$dL = d(\sigma \cdot A) = 0 \rightarrow \sigma \cdot dA + A \cdot d\sigma = 0 \qquad \text{AI.2}$$

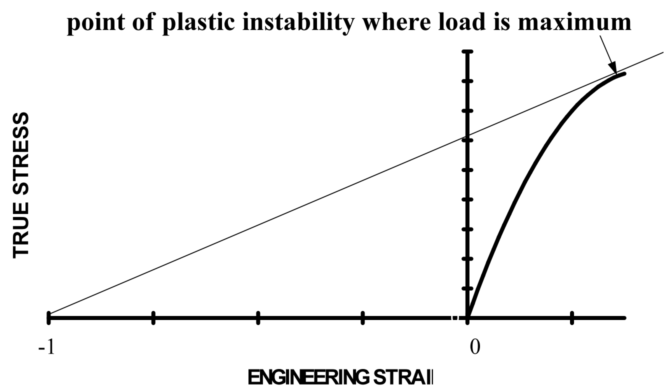
Combining Equations AI.1 and AI.2 to eliminate dA/A gives

$$\frac{d\sigma}{\sigma}=\frac{dl}{l}\rightarrow\frac{d\sigma}{\sigma}=d\bar{\varepsilon}\rightarrow\frac{d\sigma}{\sigma}=\frac{d\bar{e}}{1+\bar{e}} \qquad \text{AI.3}$$

or

$$\frac{d\sigma}{d\bar{e}}=\frac{\sigma}{1+\bar{e}} \qquad \text{AI.4}$$

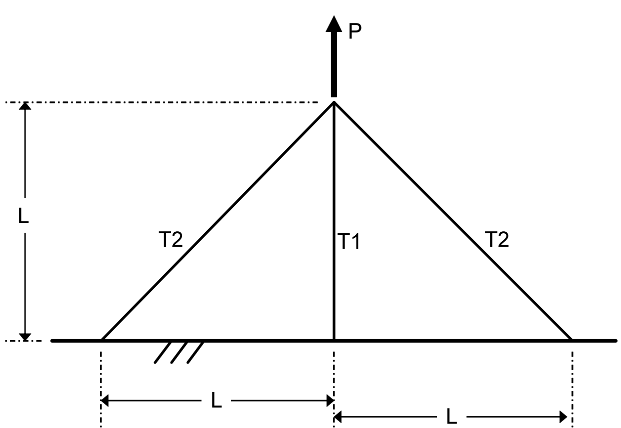
This is the condition that is fulfilled when the maximum load is reached. The construction shown in the sketch below indicates how Equation AI.4 can be solved graphically using a true stress (σ) versus engineering strain ($\bar{e}$) curve.



When the maximum load, L_{max}, is found, the ultimate tensile strength S_{ult} (an engineering stress) is given by,

$$S_{ult}=\frac{L_{max}}{A_o} \qquad \text{AI.5}$$

An Elementary Structures Problem



Small displacements, elastic-plastic bars
3 identical bars, same yield stress σ_{yp}, same area, A,
same Young's modulus, E
T1 is tensile force in center bar
T2 is tensile force in outer bars
δ1 is the change of length of the center bar
δ2 is the change of length of the outer bars
All connections are pin connections

Let,

$$T1^* = \sigma_{yp}\cdot A \qquad \delta1^* = \frac{\sigma_{yp}}{E}\cdot L \qquad \text{so that} \qquad T1^* = \frac{E\cdot A}{L}\cdot\delta1^*$$

$$T2^* = \sigma_{yp}\cdot A \qquad \delta2^* = \sqrt{2}\cdot\frac{\sigma_{yp}}{E}\cdot L \qquad T2^* = \frac{E\cdot A}{\sqrt{2}\cdot L}\cdot\delta2^*$$

Start P from zero with T1 = T2 = 0 & δ1 = δ2 = 0 and increase P monotonically.

For initial elastic region, $0 < \delta1 < \delta1^*$:

$$\delta2 = \frac{\delta1}{\sqrt{2}}$$

$$T1 = \frac{E\cdot A}{L}\cdot\delta1$$

$$T2 = \tfrac{1}{2}\cdot\frac{E\cdot A}{L}\cdot\delta1$$

$$P = T1 + \sqrt{2}\cdot T2 = \frac{1+\sqrt{2}}{\sqrt{2}}\cdot\frac{E\cdot A}{L}\cdot\delta1$$

$$\therefore P_{\text{elastic limit}} = \frac{1+\sqrt{2}}{\sqrt{2}}\cdot\frac{E\cdot A}{L}\cdot\delta1^*$$

For $\delta1 > \delta1^*$ but not enough to yield side bars:

$$T1 = T1^*$$

$$T2 = \tfrac{1}{2}\cdot\frac{E\cdot A}{L}\cdot\delta1$$

$$P = T1^* + \frac{1}{\sqrt{2}}\cdot\frac{E\cdot A}{L}\cdot\delta1$$

valid until $T2 = T2^*$, then $\delta1 = 2\cdot\delta1^*$ and,

$$P = \left(1+\sqrt{2}\right)T1^*$$

For $\delta1 > 2\cdot\delta1^*$

$$T1 = T1^*$$

$$T2 = T2^* = T1^*$$

$$P = T1^* + \sqrt{2}\cdot T2^* = \left(1+\sqrt{2}\right)T1^*$$

load does not change with increasing δ1

For unloading from $\delta 1^{**} > \sqrt{2} \cdot \delta 1^{*}$, let $\delta = \delta 1^{**} - \delta 1$

Initial unloading is elastic for $0 < \delta < 2 \cdot \delta 1^{*}$

$$T1 = T1^{*} - \frac{E \cdot A}{L} \cdot \delta$$

$$T2 = T2^{*} - \frac{1}{2} \cdot \frac{E \cdot A}{L} \cdot \delta$$

$$P = T1 + \sqrt{2} \cdot T2 \ = \left(1 + \sqrt{2}\right) T1^{*} - \frac{E \cdot A}{L} \cdot \frac{1 + \sqrt{2}}{\sqrt{2}} \cdot \delta$$

when $\delta = 2 \cdot \delta 1^{*}$, it is the end of elastic unloading and,

$$P_{\text{initial unload yield}} = -T1^{*} = -\frac{E \cdot A}{L} \cdot \delta 1^{*}$$

$$T1 = -T1^{*}$$

$$T2 = 0$$

For $2 \cdot \delta 1^{*} < \delta < 4 \cdot \delta 1^{*}$

$$T1 = -T1^{*}$$

$$T2 = -\frac{1}{2} \cdot \frac{E \cdot A}{L} \cdot \left(\delta - 2 \cdot \delta 1^{*}\right)$$

$$P = -T1^{*} - \frac{1}{\sqrt{2}} \cdot \frac{E \cdot A}{L} \cdot \left(\delta - 2 \cdot \delta 1^{*}\right)$$

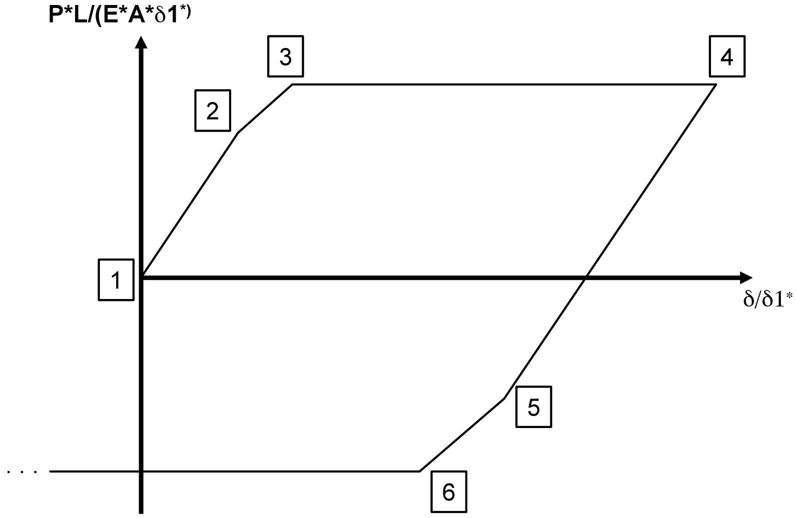
when $\delta = 4 \cdot \delta 1^{*}$, further increase in δ does not change loads

$$P = -T1^{*} - \sqrt{2} \cdot \frac{E \cdot A}{L} \cdot \delta 1^{*} = -\left(1 + \sqrt{2}\right) T1^{*}$$

$$T1 = -T1^{*}$$

$$T2 = -T1^{*}$$

The figure below shows the cycle computed above graphically.



1	(0, 0)
2	(1, 1.707)
3	(2, 2.414)
4	($\delta 1^{**}$, 2.414)
5	($\delta 1^{**}$-2, -1)
6	($\delta 1^{**}$-4, -2.414)

YIELD CONDITIONS FOR ISOTROPIC MATERIALS (IN CARTESIAN COORDINATES)

σ_{ij} = the usual spatial stress tensor, $\sigma_{ij} = \sigma_{ji}$, i = x, y, z

If the material is isotropic then the function defining the yield condition in terms of the stresses must not change when the xyz coordinates are changed. To find the admissible forms a little linear algebra follows using matrix notation,

Coordinate change to [$\bar{x}$] from [x]: $[\bar{x}] = [c] \cdot [x]$, where $[c]^{-1} = [c]^T$

Stress changes from [σ] to [$\bar{\sigma}$] where $[\bar{\sigma}] = [c] \cdot [\sigma] \cdot [c]^T$

Recall that the determinant of the product of two square matrices is equal to the product of their determinants so that $|\bar{\sigma}| = |c| \cdot |\sigma| \cdot |c^{-1}| = |\sigma|$ and the determinant is independent of the coordinate system chosen.

Now $\left| [\sigma] - \lambda \cdot [I] \right|$ is the determinant of a legitimate stress field where λ is to be determined. In fact, when this determinant is set to zero, the three roots for λ are the principal stresses which are, of course, independent of the coordinate system chosen. The cubic equation in λ resulting from the expansion of this determinant, DET, is,

DET =

$$-\lambda^3 + \left(\sigma_{xx} + \sigma_{yy} + \sigma_{zz}\right)\lambda^2 + \left(-\sigma_{xx} \cdot \sigma_{yy} - \sigma_{yy} \cdot \sigma_{zz} - \sigma_{zz} \cdot \sigma_{xx} + \sigma_{yz} \cdot \sigma_{zy} + \sigma_{zx} \cdot \sigma_{xz} + \sigma_{xy} \cdot \sigma_{yx}\right)\lambda$$
$$+ (\sigma_{xx} \cdot \sigma_{yy} \cdot \sigma_{zz} + \sigma_{yx} \cdot \sigma_{yz} \cdot \sigma_{zx} + \sigma_{xz} \cdot \sigma_{zy} \cdot \sigma_{yz} - \sigma_{zx} \cdot \sigma_{xz} \cdot \sigma_{yy} - \sigma_{zy} \cdot \sigma_{yz} \cdot \sigma_{xx} - \sigma_{zz} \cdot \sigma_{xy} \cdot \sigma_{yx})$$

Since the roots for λ are independent of the coordinate system chosen, the coefficients, called invariants of the stress tensor, must also be independent. Three stress functions, IN1, IN2 and IN3 and, J2, a combination of IN1 and IN2 are given below,

$$\mathrm{IN1} = \sigma_{xx} + \sigma_{yy} + \sigma_{zz}$$
$$\mathrm{IN2} = -\sigma_{xx} \cdot \sigma_{yy} - \sigma_{yy} \cdot \sigma_{zz} - \sigma_{zz} \cdot \sigma_{xx} + \sigma_{yz} \cdot \sigma_{zy} + \sigma_{zx} \cdot \sigma_{xz} + \sigma_{xy} \cdot \sigma_{yx}$$
$$\mathrm{IN3} = \sigma_{xx} \cdot \sigma_{yy} \cdot \sigma_{zz} + \sigma_{yx} \cdot \sigma_{yz} \cdot \sigma_{zx} + \sigma_{xz} \cdot \sigma_{zy} \cdot \sigma_{yz} - \sigma_{zx} \cdot \sigma_{xz} \cdot \sigma_{yy} - \sigma_{zy} \cdot \sigma_{yz} \cdot \sigma_{xx} - \sigma_{zz} \cdot \sigma_{xy} \cdot \sigma_{yx}$$
$$\mathrm{J2} = \mathrm{IN2} + \tfrac{1}{3} \cdot (\mathrm{IN1})^2 = \tfrac{1}{3} \cdot \left(\sigma_{xx}{}^2 + \sigma_{yy}{}^2 + \sigma_{zz}{}^2 - \sigma_{xx} \cdot \sigma_{yy} - \sigma_{yy} \cdot \sigma_{zz} - \sigma_{zz} \cdot \sigma_{xx}\right) + \sigma_{zx}{}^2 + \sigma_{zy}{}^2 + \sigma_{yz}{}^2$$

A very popular criterion for the onset of yielding, called the von Mises yield condition, is,

$$\mathrm{J2} = k^2$$

where k is the yield point in *shear*. The relationship between the yield point in shear and the yield point in tension, σ_{yp}, is easily found by taking σ_{xx} as the only nontrivial stress and substituting it into the last equation to obtain,

$$\sigma_{yp} = \sqrt{3} \cdot k$$

Comparison of the Beltrami and the Von Mises Yield Criteria

The von Mises criterion may be written, in a general Cartesian coordinate system, as,

$$\sigma_{YP}^2 = \sigma_X^2 + \sigma_Y^2 + \sigma_Z^2 - \sigma_X \cdot \sigma_Y - \sigma_Y \cdot \sigma_Z - \sigma_Z \cdot \sigma_X + 3 \cdot \left(\tau_{XY}^2 + \tau_{YZ}^2 + \tau_{ZX}^2\right)$$

where σ is a normal stress and τ is a shear stress. Consider the case of a long, closed cylinder with internal pressure of p_i and no external pressure. Superpose an axial load, P, on the closed cylinder. The elastic stress state away from the ends of the cylinder and at the inner radius is,

$$\sigma_R = \text{radial stress} = -\,p_i$$

$$\sigma_\theta = \text{circumferential stress} = p_i \cdot \frac{OD^2 + ID^2}{OD^2 - ID^2}$$

$$\sigma_Z = \text{axial stress} = p_i \cdot \frac{ID^2}{OD^2 - ID^2} + \frac{P}{\frac{\pi}{4} \cdot \left(OD^2 - ID^2\right)}$$

and the shear stresses vanish. In this case the von Mises criterion becomes,

$$\left(OD^2 - ID^2\right)^2 \cdot \sigma_{YP}^2 = 3 \cdot OD^4 \cdot p_i^2 + \frac{P^2}{\left(\frac{\pi}{4}\right)^2}$$

so that.

$p_{iy}\big|_{P=0}$ = internal pressure to initiate yielding when P = 0 $= \dfrac{\sigma_{YP} \cdot \left(OD^2 - ID^2\right)}{\sqrt{3} \cdot OD^2}$

$P_y\big|_{p_i=0}$ = axial load to initiate yielding when p_i = 0 $= \frac{\pi}{4} \cdot \sigma_{YP} \cdot \left(OD^2 - ID^2\right)$

and the criterion may be written as,

$$\left(\frac{p_i}{p_{iy}\big|_{P=0}}\right)^2 + \left(\frac{P}{P_y\big|_{p_i=0}}\right)^2 = 1$$

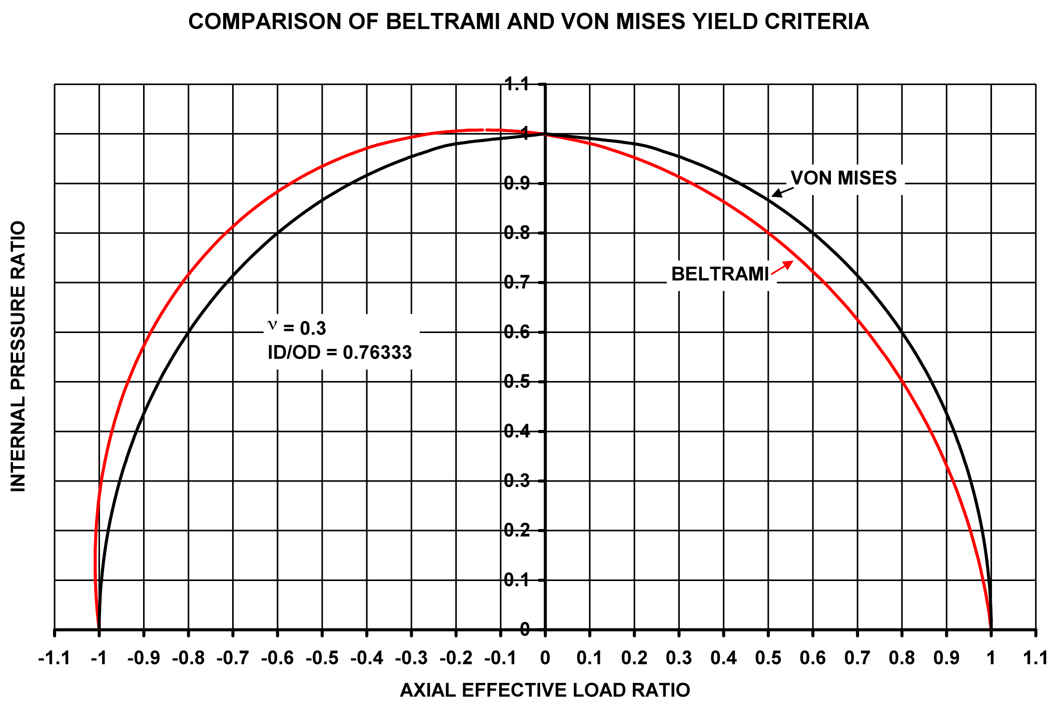
The same procedure will now be followed for the Beltrami yield criterion. It may be written as,

$$\sigma_{YP}^2 = \left(\sigma_X + \sigma_Y + \sigma_Z\right)^2 - 2 \cdot (1 + \nu) \cdot \left(\sigma_X \cdot \sigma_Y + \sigma_Y \cdot \sigma_Z + \sigma_Z \cdot \sigma_X - \tau_{XY}^2 - \tau_{YZ}^2 - \tau_{ZX}^2\right)$$

where ν is Poisson’s ratio. Note that the von Mises and Beltrami criteria coincide when ν = 0.5. When the stress state at the ID of the cylinder is substituted into this criterion, the result is,

$$\left(\frac{P}{P_y|_{p_i=0}}\right)^2 + \left(2\cdot\sqrt{3} - \frac{4}{\sqrt{3}}\cdot(1+\nu)\right)\cdot\left(\frac{ID}{OD}\right)^2\cdot\frac{p_i}{p_{iy}|_{P=0}}\cdot\frac{P}{P_y|_{p_i=0}}$$
$$+\left(\frac{p_i}{p_{iy}|_{P=0}}\right)^2\cdot\left(3\cdot\left(\frac{ID}{OD}\right)^4 - \frac{2}{3}\cdot(1+\nu)\cdot\left(3\cdot\left(\frac{ID}{OD}\right)^4 - 1\right)\right) - 1 = 0$$

The figure below gives a comparison of the two criteria when $\nu = 0.3$ and the diameter ratio is 0.76333.



REFERENCE:

A quote from History of Strength of Materials by Stephen P. Timoshenko:

"Beltrami[1] suggested that, in determining the critical values of combined stresses, the amount of strain energy stored per unit volume of the material should be adopted as the criterion of failure. This theory does not agree with experiments, however, since a large amount of strain energy may be stored in a material under uniform hydrostatic pressure without the onset of fracture or yielding,"

1 Rendiconti, p. 704, 1885; *Math. Ann.*, p. 94, 1903.

From *Theory of Elasticity, Third Edition*, by S. P. Timoshenko and J. N. Goodier:

Define stress invariants I_1, I_2 and I_3 as

$I_1 = \sigma_x + \sigma_y + \sigma_z$

$$I_2 = \sigma_x \cdot \sigma_y + \sigma_y \cdot \sigma_z + \sigma_z \cdot \sigma_x - \tau_{xy}^2 - \tau_{yz}^2 - \tau_{zx}^2$$

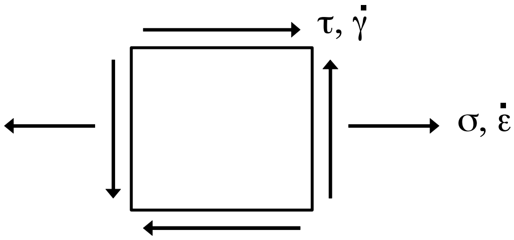
$$I_3 = \sigma_x \cdot \sigma_y \cdot \sigma_z + 2 \cdot \tau_{xy} \cdot \tau_{yz} \cdot \tau_{zx} - \sigma_x \cdot \tau_{yz}^2 - \sigma_y \cdot \tau_{zx}^2 - \sigma_z \cdot \tau_{xy}^2$$

then the strain energy per unit volume, V_0, is given by,

$$V_0 = \frac{1}{2 \cdot E} \cdot \left[I_1^2 - 2 \cdot (1 + \nu) \cdot I_2 \right]$$

SIMPLE PROBLEM, PRANDTL-REUSS MATERIAL, INCOMPRESSIBLE

The formulation of the constitutive equations used in Prager & Hodges' treatise is somewhat involved and not done very well from a Physics point of view in their book. A different formulation is presented later. Appendix 5 presents the equations used by Prager & Hodge. The reader is asked to accept Appendix 5 so that problems can be worked as soon as possible. Note that λ is a new unknown that must be determined using the yield condition. The first problem is shown below,



1. Apply σ first to $\sqrt{3}\cdot k$, no plastic flow.
2. Keep $\dot{\varepsilon} = 0$ and apply τ.

For second loading:

Rate of doing work = $\dot{W} = \tau \cdot \dot{\gamma}$

Parameter in Prandtl-Reuss equation = $\lambda = \dfrac{G\cdot \dot{W}}{k^2} = \dfrac{G\cdot\tau\cdot\dot{\gamma}}{k^2}$

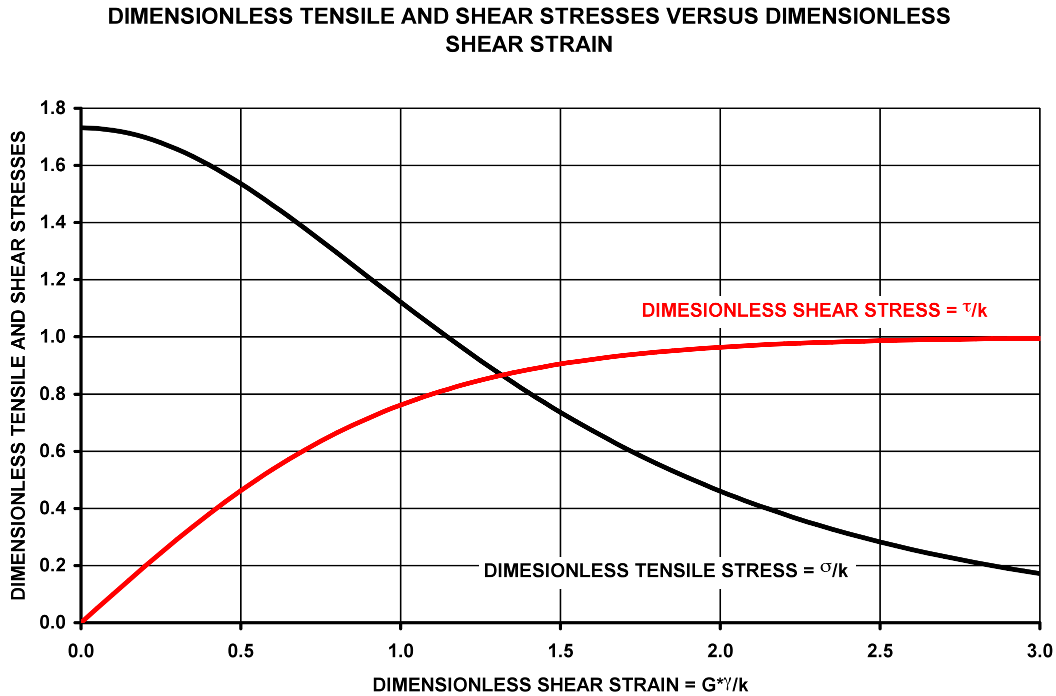
Prandtl-Reuss equation: $$\dot{\tau} = G\cdot\left(\dot{\gamma} - \frac{\tau\cdot\dot{\gamma}}{k^2}\cdot\tau\right) = G\cdot\dot{\gamma}\cdot\left(1-\frac{\tau^2}{k^2}\right) \rightarrow \frac{G\cdot\dot{\gamma}}{k} = \frac{\left(\frac{\dot{\tau}}{k}\right)}{1-\frac{\tau^2}{k^2}}$$

Use $$\int\frac{dx}{p^2 - x^2} = \frac{1}{p}\cdot\tanh^{-1}\left(\frac{x}{p}\right)$$

To obtain $$\frac{G\cdot\gamma}{k} = \tanh^{-1}\left(\frac{\tau}{k}\right) \rightarrow \frac{\tau}{k} = \tanh\left(\frac{G\cdot\gamma}{k}\right)$$

Use yield criterion $$\sigma^2 + 3\cdot\tau^2 = 3\cdot k^2 \rightarrow \frac{\sigma}{k} = \sqrt{3}\cdot\sqrt{1-\frac{\tau^2}{k^2}} = \frac{\sqrt{3}}{\cosh\left(\frac{G\gamma}{k}\right)}$$

See figure below


DIMENSIONLESS TENSILE AND SHEAR STRESSES VERSUS DIMENSIONLESS SHEAR STRAIN
DIMENSIONLESS TENSILE AND SHEAR STRESSES
DIMESIONLESS SHEAR STRESS = τ/k
DIMESIONLESS TENSILE STRESS = σ/k
DIMENSIONLESS SHEAR STRAIN = G*γ/k
0.0
0.2
0.4
0.6
0.8
1.0
1.2
1.4
1.6
1.8
0.0
0.5
1.0
1.5
2.0
2.5
3.0

SECOND SIMPLE PROBLEM, PRANDTL-REUSS MATERIAL, INCOMPRESSIBLE

Similar to first but order of loading is reversed.

1. Apply τ up to k with no plastic flow.
2. Hold $\dot{\gamma} = 0$ and apply σ.

After first step: $\sigma = 0$, $\tau = k$, $\varepsilon = 0$ and $\gamma = \dfrac{k}{G}$

For second step: $\dot{W} = \sigma \cdot \dot{\varepsilon}\,,\quad \dot{\tau} = G \cdot \dfrac{\dot{W}}{k^2} \cdot \tau\,,\quad \dot{\sigma} = 3 \cdot G \cdot \left(\dot{\varepsilon} - \dfrac{\sigma \cdot \dot{\varepsilon}}{3 \cdot k^2} \cdot \sigma \right) \;\rightarrow\; \dfrac{\dot{\sigma}}{3 \cdot G} = \left(1 - \dfrac{\sigma^2}{3 \cdot k^2} \right) \cdot \dot{\varepsilon}$

Integrating: $\dfrac{\sigma}{\sqrt{3} \cdot k} = \tanh\left(\dfrac{\sqrt{3} \cdot G}{k} \cdot \varepsilon \right)$

Use yield criterion $\frac{1}{3} \cdot \sigma^2 + \tau^2 = k^2 \;\rightarrow\; \dfrac{\tau}{k} = \dfrac{1}{\cosh\left(\dfrac{\sqrt{3} \cdot G}{k} \cdot \varepsilon \right)}$

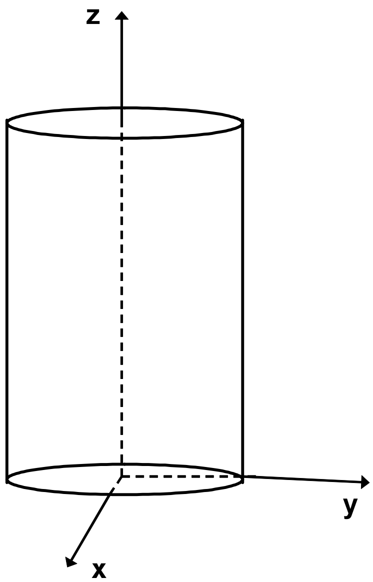
Comparison of the solutions for the two simple problems,

First problem $\dfrac{\sigma}{\sqrt{3} \cdot k} = \dfrac{1}{\cosh\left(\dfrac{G \cdot \gamma}{k} \right)}\,,\quad \dfrac{\tau}{k} = \tanh\left(\dfrac{G \cdot \gamma}{k} \right)$

Second problem $\dfrac{\sigma}{\sqrt{3} \cdot k} = \tanh\left(\dfrac{\sqrt{3} \cdot G \cdot \varepsilon}{k} \right)\,,\quad \dfrac{\tau}{k} = \dfrac{1}{\cosh\left(\dfrac{\sqrt{3} \cdot G \cdot \varepsilon}{k} \right)}$

Note that $\dfrac{\sigma}{\sqrt{3} \cdot k}$ and $\dfrac{\tau}{k}$ as well as $\dfrac{\sqrt{3} \cdot G \cdot \varepsilon}{k}$ and $\dfrac{G \cdot \gamma}{k}$ switch roles in the two solutions. More important, the solutions show that the results are load path dependent as first step can induce a plastic strain

A LESS SIMPLE PROBLEM SOLVED USING AN ANALOGY, TORSION



Right cylinder with z axis intersecting the centroid of the cross section

The cross section is simply connected (enclosed by single curve)

Axial length of cylinder is much greater than the cross-section extent

Neglect end effects

An axial torque, T, is applied to the top of the cylinder

An axial torque, -T, is applied to the bottom of the cylinder

θ = angle of twist per unit length away from ends

The lateral surfaces of the cylinder are stress-free

.

Assume the displacements in the elastic *and* plastic regions of each cross section are,

$$u_x = -y\cdot z\cdot\theta$$
$$u_y = x\cdot z\cdot\theta$$
$$w = w(x, y : \theta)$$

so that,

$$\varepsilon_x = \varepsilon_y = \varepsilon_z = \gamma_{xy} = 0$$
$$\gamma_x \equiv \gamma_{zx} = -\theta\cdot y + w,_x$$
$$\gamma_y \equiv \gamma_{yz} = \theta\cdot x + w,_y$$
$$\tau_x \equiv \tau_{xz}$$
$$\tau_y \equiv \tau_{yz}$$

Now observe that,

$$w,_{xy} = \gamma_x,_y + \theta \quad \text{and} \quad w,_{xy} = \gamma_y,_x - \theta$$

so that,

$$\gamma_y,_x - \gamma_x,_y = 2\cdot\theta$$

The only non-vanishing equilibrium equation is for the z direction and it becomes,

$$\tau_{x},_{x} + \tau_{y},_{y} = 0$$

The function, ψ, can be introduced to represent the two stresses, τ_x and τ_y, with a single function as,

$$\tau_x = \psi(x, y : \theta),_y \quad \text{and} \quad \tau_y = -\psi(x, y : \theta),_x$$

Note that, when the stresses are defined this way, the equilibrium equation above is automatically satisfied. The boundary conditions on the lateral surface are,

$$X = \sigma_x \cdot \cos(\alpha) + \tau \cdot \sin(\alpha)$$
$$Y = \sigma_y \cdot \sin(\alpha) + \tau \cdot \cos(\alpha)$$

where α is the inclination of the outward unit length boundary normal vector to the x axis so that on the boundary progressing in a counter-clockwise direction,

$$dx = -ds \cdot \sin(\alpha) \quad \text{and} \quad dy = ds \cdot \cos(\alpha)$$

The shear stress at the boundary must be in the direction of the tangent to the boundary so,

$$\tau_x \cdot \cos(\alpha) + \tau_y \cdot \sin(\alpha) = \tau_x \cdot \frac{dy}{ds} - \tau_y \cdot \frac{dx}{ds} = \psi,_y \cdot \frac{dy}{ds} + \psi,_x \cdot \frac{dx}{ds} = \psi,_s = 0$$

Consequently, ψ is constant on the simply connected boundary. The constant is usually set to zero and that will be done here. The net forces, Sx and Sy, and net torque, T, on the cross section are,

$$Sx = \int_A \tau_x \cdot dA = \int_A \psi,_y \cdot dA = \oint_s \psi \cdot n_y \cdot ds = 0$$
$$Sy = \int_A \tau_y \cdot dA = -\int_A \psi,_x \cdot dA = -\oint_s \psi \cdot n_x \cdot ds = 0$$
$$T = \int_A (x \cdot \tau_y - y \cdot \tau_x) \cdot dA = -\int_A \left[(x \cdot \psi),_x - \psi + (y \cdot \psi),_y - \psi\right] dA = -\oint_s \psi \cdot (x \cdot n_x + y \cdot n_y) \cdot ds + 2 \cdot \int_A \psi \cdot dA$$
$$= 2 \cdot \int_A \psi \cdot dA = 2 \cdot (\text{volume under } \psi \text{ surface})$$

The results given above are valid for both the elastic and the plastic regions of the cross section. The elastic case is considered next. Hooke's law gives,

$$\gamma_x = \frac{1}{G} \cdot \tau_x = \frac{1}{G} \cdot \psi,_y \quad \text{and} \quad \gamma_y = \frac{1}{G} \cdot \tau_y = -\frac{1}{G} \cdot \psi,_x$$

When these results are substituted into $\gamma_{y},_{x} - \gamma_{x},_{y} = 2 \cdot \theta$, the result is,

$$\nabla^2 \psi \equiv \psi,_{xx} + \psi,_{yy} = -2 \cdot G \cdot \theta$$

This is the governing equation for the elastic problem. To illustrate this, take the familiar case of a solid circular rod of radius a. Let r be the radial coordinate so that,

$$\psi = C\cdot\left(a^2 - x^2 - y^2\right) = C\cdot\left(a^2 - r^2\right)$$

$$\tau_x = \psi,_y = -2\cdot C\cdot y\,,\quad \tau_y = -\psi,_x = 2\cdot C\cdot x$$

$$\nabla^2\psi = -4\cdot C = -2\cdot G\cdot\theta \quad \rightarrow \quad C = \tfrac{1}{2}\cdot G\cdot\theta$$

$$\tau_x = -G\cdot\theta\cdot y\,,\quad \tau_y = G\cdot\theta\cdot x\,,\quad \tau\Big|_{\text{resolved}} = \sqrt{{\tau_x}^2 + {\tau_y}^2} = G\cdot\theta\cdot r$$

$$T = 2\cdot\int_{r=0}^{r=a} \tfrac{1}{2}\cdot G\cdot\theta\cdot\left(a^2 - r^2\right)\cdot 2\cdot\pi\cdot r\, dr = \tfrac{1}{2}\cdot\pi\cdot G\cdot\theta\cdot a^4$$

Therefore,

$$G\cdot\theta = \frac{2\cdot T}{\pi\cdot a^4}$$

and the limit for θ in the elastic region is when the resolved shear stress at $r = a$ equals k giving,

$$T\Big|_{\text{elastic limit}} = \tfrac{1}{2}\cdot\pi\cdot k\cdot a^3\,,\quad \theta\Big|_{\text{elastic limit}} = \frac{k}{G\cdot a}$$

This solution is easily extended beyond the elastic region by setting a "roof" on the circular cross section whose radial slope is equal to k. The ψ surface is limited to contact the cone but it cannot penetrate the cone. In the limit the torque in the fully plastic state is proportional to the volume under the full cone so that,

$$T\Big|_{\text{fully plastic}} = 2\cdot(\text{volume of cone})$$

$$(\text{volume of cone}) = \tfrac{1}{3}\cdot(\text{area of base})\cdot(\text{altitude})$$

$$T\Big|_{\text{fully plastic}} = 2\cdot\tfrac{1}{3}\left(\pi\cdot a^2\right)(k\cdot a) = \tfrac{2}{3}\cdot\pi\cdot k\cdot a^3$$

Then,

$$\frac{T\Big|_{\text{fully plastic}}}{T\Big|_{\text{elastic limit}}} = \tfrac{4}{3}$$

P&H give the solution for torques between $T\Big|_{\text{elastic limit}}$ and $T\Big|_{\text{fully plastic}}$ in terms of the angle of twist, θ, as,

$$T = \tfrac{2}{3}\cdot\pi\cdot a^3\cdot k\cdot\left(1 - \frac{1}{4\cdot a^3}\cdot\left(\frac{k}{G\cdot\theta}\right)^3\right) \qquad \theta > \frac{k}{G\cdot a}$$

The above covers the stress determinations. P&H give several solutions to torsion problems.

Now turn attention to the determination of $w(x,y:\theta)$, the warping function. Suppose the stress solution described above has been found so that γ_x and γ_y may be regarded as known everywhere in the cross section. The elastic and plastic regions are bounded by C.

First consider the determination of $w(x,y:\theta)$ in the elastic region. In this region,

$$w,_x = \theta \cdot y + \frac{\tau_x}{G}$$

$$w,_y = -\theta \cdot x + \frac{\tau_y}{G}$$

so that,

$$\frac{dw}{ds} = w,_x \cdot \frac{dx}{ds} + w,_y \cdot \frac{dy}{ds}$$

When w is specified at one point, P_O, in the elastic region, the value of w can be found anywhere in the elastic region by integrating the last equation. The integration is path independent since displacement compatibility is satisfied. Therefore, w can be found on the elastic-plastic boundary. Note that in the elastic region,

$$\frac{\gamma_x}{\gamma_y} = \frac{\tau_x}{\tau_y}$$

Consider a point, P, in the elastic region. When the elastic-plastic boundary, Γ, reaches P, then the yield criterion is satisfied at P so that τ = resolved shear stress = $\sqrt{\tau_x^{\,2} + \tau_y^{\,2}}$ and it is directed parallel to a point on the boundary, C, curve. This point is found as the point on C whose normal (to C) passes through P. The stresses at point P do not change with further twisting.

In the plastic region,

$$G \cdot \dot{\gamma}_x = \dot{\tau}_x + \lambda \cdot \tau_x$$

$$G \cdot \dot{\gamma}_y = \dot{\tau}_y + \lambda \cdot \tau_y$$

but $\dot{\tau}_x = \dot{\tau}_y = 0 \quad \rightarrow \quad G \cdot \dot{\gamma}_x = \lambda \cdot \tau_x$ & $G \cdot \dot{\gamma}_y = \lambda \cdot \tau_y$ so that,

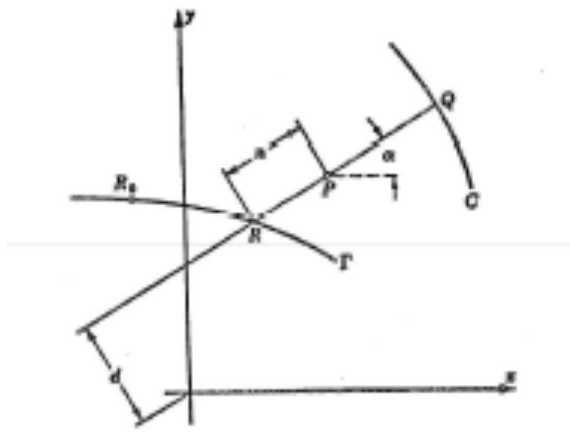
$\dfrac{\dot{\gamma}_x}{\dot{\gamma}_y} = \dfrac{\tau_x}{\tau_y} = \text{constant} \quad \therefore \; \gamma_x = \dfrac{\tau_x}{\tau_y} \cdot \gamma_y + c\,, \quad c = 0$ and in the plastic region,

$$\frac{\gamma_x}{\gamma_y} = \frac{\tau_x}{\tau_y}$$

Now the values of w in the plastic region may be found by solving the equations below under the condition that $k = \sqrt{\tau_x^{\,2} + \tau_y^{\,2}}$,

$$w,_x = \theta \cdot y + \frac{\lambda \cdot \tau_x}{G}$$
$$w,_y = -\theta \cdot x + \frac{\lambda \cdot \tau_y}{G}$$

A clever way to accomplish this is indicated in the figure below.



On RQ $\tau_x = -k \cdot \sin(\alpha)$ and $\tau_y = k \cdot \cos(\alpha)$ so that,

$$w,_x = \theta \cdot y - \frac{\lambda \cdot k}{G} \cdot \sin(\alpha)$$
$$w,_y = -\theta \cdot x + \frac{\lambda \cdot k}{G} \cdot \cos(\alpha)$$

Combining these two equations to eliminate $\frac{\lambda \cdot k}{G}$ yields,

$$w,_x \cdot \cos(\alpha) + w,_y \cdot \sin(\alpha) = \frac{dw}{dn} = \theta \cdot (y \cdot \cos(\alpha) - x \cdot \sin(\alpha)) = \theta \cdot d \quad \text{or} \quad \frac{dw}{dn} = \theta \cdot d$$

P&H CHAPTER 4, PLANE STRAIN, AXIAL SYMMETRY

The solution presented here differs from the standard Lame' solutions found in all elasticity textbooks in two ways which are,

1. The material is incompressible (both elastic and plastic strains, $\therefore \varepsilon_{ij} = e_{ij}$).
2. The elastic-plastic solution is found.

Take symmetry into account so that in both the elastic and plastic regions,

r = radial coordinate measured from center of hollow cylinder.
u = outward radial displacement, the only non-trivial displacement, a function of r only

$$\varepsilon_r = u_{,r} \qquad \gamma_{r\theta} = 0$$
$$\varepsilon_\theta = \frac{u}{r} \qquad \gamma_{\theta z} = 0$$
$$\varepsilon_z = 0 \qquad \gamma_{zr} = 0$$

Note that $\varepsilon_r:\varepsilon_\theta:\varepsilon_z = e_r:e_\theta:e_z = 1:-1:0$. The incompressibility condition yields,

$$\frac{du}{dr} + \frac{u}{r} = 0 \quad \rightarrow \quad u = \frac{D}{r}$$

and

$$\varepsilon_r = -\frac{D}{r^2}, \quad \varepsilon_\theta = \frac{D}{r^2}$$

In the elastic region,

$$s_r = -2 \cdot G \cdot \frac{D}{r^2}, \quad s_\theta = G \cdot \frac{D}{r^2}, \quad s_z = 0$$

Denote the mean stress by s so that,

$$\sigma_r = -2 \cdot G \cdot \frac{D}{r^2} + s, \quad \sigma_\theta = G \cdot \frac{D}{r^2} + s, \quad s_z = s$$

The radial and tangential force equilibrium conditions must be satisfied and they are,

$$\sigma_{r,r} + \frac{1}{r} \cdot \tau_{r\theta,\theta} + \frac{\sigma_r - \sigma_\theta}{r} = 0, \quad \frac{1}{r} \cdot \sigma_{\theta,\theta} + \tau_{r\theta,r} + \frac{2}{r} \cdot \tau_{r\theta} = 0$$

In this case the equations are satisfied so long as s is a constant. This is consistent with the notion that a hydrostatic pressure will not alter the displacement solution for an isotropic, incompressible material. In order to determine the elastic-plastic boundary the yield condition is required. It is,

$$J2 = \tfrac{1}{2}\cdot s_r^{\,2} + \tfrac{1}{2}\cdot s_\theta^{\,2} = 4\cdot G^2\cdot\frac{D^2}{r^4} = k^2$$

$$k = 2\cdot G\cdot\frac{D}{r^2}$$

Since k decreases with r, it is clear that the plastic region (if there is one) lies between the inner radius, a, of the cylinder and the elastic-plastic boundary at $r = \rho$. The elastic region is between r = b and the outer radius, b, of the cylinder. The last equation is satisfied at $r = \rho$ so that,

$$D = \frac{k\cdot\rho^2}{2\cdot G}$$

and the condition that $\sigma_r = 0$ at r = b then gives,

$$s = \frac{2\cdot G\cdot D}{b^2} = k\cdot\frac{\rho^2}{b^2}$$

The solution in the elastic region ($\rho < r < b$) is,

$$u = \frac{k\rho^{\;2}}{2\cdot G\cdot r}, \quad \text{tangential and axial displacements} = 0$$

$$\sigma_{r\theta} = k\cdot\frac{\rho^2}{b^2}\cdot\left(1-\frac{b^2}{r^2}\right), \quad \sigma_{\,z} = k\cdot\frac{\rho^2}{b^2}\cdot\left(1+\frac{b^2}{r^2}\right), \quad \sigma\;\; = k\cdot\frac{\rho^2}{b^2}$$

For the plastic region, recall that $e_r:e_\theta:e_z$ = 1:-1:0 and, since all strains start at zero in the unloaded state, $\dot{e}_r:\dot{e}_\theta:\dot{e}_z = 1:-1:0$ so from the inner radius, a, to $r = \rho$,

$$s_r:s_\theta:s_z = \dot{s}_r:\dot{s}_\theta:\dot{s}_z = 1:-1:0$$

This implies that,

$$s_r = -k, \quad s_\theta = k, \quad s_z = 0$$

and then in the plastic region,

$$\sigma_r = s-k, \quad \sigma_\theta = s+k, \quad \sigma_z = s$$

Now consider the radial force equilibrium equation,

$$\frac{d\sigma_r}{dr} + \frac{\sigma_r-\sigma_\theta}{r} = 0 \quad\rightarrow\quad \frac{ds}{dr} = \frac{2\cdot k}{r} \quad\rightarrow\quad s = 2\cdot k\cdot\ln(r) + f(\rho)$$

Continuity for σ_r at $r = \rho$ (the elastic-plastic boundary) gives,

$$f(\rho) = k \cdot \left(\frac{\rho^2}{b^2} - 2 \cdot \ln(\rho) \right)$$

$$s = k \cdot \left(\frac{\rho^2}{b^2} + 2 \cdot \ln\left(\frac{r}{\rho} \right) \right)$$

$$\sigma_r = -k \cdot \left(1 - \frac{\rho^2}{b^2} - 2 \cdot \ln\left(\frac{r}{\rho} \right) \right)$$

$$\sigma_\theta = k \cdot \left(1 + \frac{\rho^2}{b^2} + 2 \cdot \ln\left(\frac{r}{\rho} \right) \right)$$

$$\sigma_z = k \cdot \left(\frac{\rho^2}{b^2} + 2 \cdot \ln\left(\frac{r}{\rho} \right) \right)$$

The internal pressure, p, applied at r = a, is equal to $-\sigma_r\big|_{r=a}$ so that,

$$p = k \cdot \left(1 - \frac{\rho^2}{b^2} - 2 \cdot \ln\left(\frac{a}{\rho} \right) \right)$$

Now,

$$p^* = \text{initiation of plastic flow} = p\big|_{\rho = a} = k \cdot \left(1 - \frac{a^2}{b^2} \right)$$

$$p^{**} = \text{fully plastic flow} = p\big|_{\rho = b} = 2 \cdot k \cdot \ln\left(\frac{b}{a} \right)$$

P&H has a careful coverage for this problem. This includes,

1. Comparison of this solution with Nadai's compressible material solution
2. Unloading from $p^* < p < p^{**}$. If $\frac{b}{a} \le 2.22$ the unloading will always be elastic.
3. The residual stress state can be used to determine the shake-down condition as,

   $p < p^{**}$ for $\frac{b}{a} \le 2.22$

   $p < 2 \cdot p^*$ for $\frac{b}{a} \ge 2.22$
4. The case of unrestricted plastic flow is examined using plane strain and Mises theory (elastic strains neglected). The solution shows that for $p > p^{**}$ the pressure decreases. Therefore, in the classical sense, the solution is unstable.
5. The strains can be large enough to become "finite" strains. Section 18 of P&H gives a very good discussion on this point. The conclusion is "finite" strains need not be considered when using Mises theory but Prandtl-Reuss theory requires they be taken into account if the strains become sufficiently large.

P&H CHAPTER 5, PLANE STRAIN, GENERAL THEORY

Shear Lines in Plane Strain

For this chapter only Mises theory (neglect elastic strains) is used and there are no body forces.

Nothing varies in the z direction, w = 0, u = u(x,y) and v = v(x,y) so that the only non-vanishing strains are,

$$\dot{\varepsilon}_x = u,_x \ , \quad \dot{\varepsilon}_y = v,_y \ , \quad \dot{\gamma} = u,_y + v,_x$$

The non-trivial Mises theory constitutive equations are,

$$2 \cdot G \cdot \dot{\varepsilon}_x = \lambda \cdot s_x$$
$$2 \cdot G \cdot \dot{\varepsilon}_y = \lambda \cdot s_y$$
$$s_z = 0$$
$$G \cdot \dot{\gamma} = \lambda \cdot \tau$$

Let σ be the mean stress so the condition $s_z = 0$ gives,

$$\sigma = \sigma_z = \tfrac{1}{2} \cdot \left(\sigma_x + \sigma_y\right)$$
$$s_x = \tfrac{1}{2} \cdot \left(\sigma_x - \sigma_y\right)$$
$$s_y = -\tfrac{1}{2} \cdot \left(\sigma_x - \sigma_y\right)$$

The von Mises yield condition for these circumstances is,

$$\tfrac{1}{2} \cdot {s_x}^2 + \tfrac{1}{2} \cdot {s_y}^2 + \tau^2 = \tfrac{1}{4} \cdot \left(\sigma_x - \sigma_y\right)^2 + \tau^2 = k^2$$

This form suggests the Mohr's circle construction shown in the formula sheet, Appendix 4. With the definitions,

$$\omega \equiv \frac{\sigma_x + \sigma_y}{4 \cdot k} = \frac{\sigma}{2 \cdot k}$$
$$\theta \equiv \tfrac{1}{2} \cdot \arctan\left(\frac{2 \cdot \tau}{\sigma_x - \sigma_y}\right) + \tfrac{\pi}{4}$$

The variables σ_x, σ_y and τ and the yield condition are reduced to two variables, ω and θ, so that,

$$\sigma_x = 2 \cdot k \cdot \omega + k \cdot \sin(2 \cdot \theta)$$
$$\sigma_x = 2 \cdot k \cdot \omega - k \cdot \sin(2 \cdot \theta)$$
$$\tau = -k \cdot \cos(2 \cdot \theta)$$

ω is a dimensionless mean stress while θ is an angle in the physical plane. The x face is a surface with a constant value of x on that face. The angle θ is the angle the x face must be rotated in a counter clockwise direction to reach the face on which the maximum shear stress acts. The formula sheet contains a sketch showing the various faces and the stresses acting upon them.

The stresses plotted for Mohr's circle are N_α and T_α given in the formula sheet. After a few trigonometric manipulations these expressions become,

$$N_\alpha = \tfrac{1}{2}\cdot\left(\sigma_x + \sigma_y\right) + \tfrac{1}{2}\cdot\left(\sigma_x - \sigma_y\right)\cos(2\cdot\alpha) + \tau\cdot\sin(2\cdot\alpha)$$
$$T_\alpha = \tfrac{1}{2}\cdot\left(\sigma_x - \sigma_y\right)\sin(2\cdot\alpha) - \tau\cdot\cos(2\cdot\alpha)$$

It is easily verified that the Mohr circle construction shown on the formula sheet is consistent with the above equations. The radius of Mohr's circle in the plastic region is computed as,

$$\text{Radius} = \left(N_\alpha - \frac{\sigma_x + \sigma_y}{2}\right)^2 + {T_\alpha}^2 = \tfrac{1}{4}\cdot\left(\sigma_x + \sigma_y\right)^2 + \tau^2 = k^2$$

When the stresses in terms of ω and θ are substituted into the force equilibrium equations, the result is,

$$\omega,_x + \theta,_x\cdot\cos(2\cdot\theta) + \theta,_y\cdot\sin(2\cdot\theta) = 0$$
$$\omega,_y - \theta,_y\cdot\cos(2\cdot\theta) + \theta,_x\cdot\sin(2\cdot\theta) = 0$$

Consider a differential length line segment in the x-y plane. The changes along the line (along s) of ω and θ are given by,

$$\omega,_s = \omega,_x\cdot x,_s + \omega,_y\cdot y,_s\,, \quad \theta,_s = \theta,_x\cdot x,_s + \theta,_y\cdot y,_s$$

The last four equations can be written in matrix form as,

$$\begin{bmatrix} x,_s & y,_s & 0 & 0 \\ 0 & 0 & x,_s & y,_s \\ 1 & 0 & \cos(2\cdot\theta) & \sin(2\cdot\theta) \\ 0 & 1 & \sin(2\cdot\theta) & -\cos(2\cdot\theta) \end{bmatrix} \cdot \begin{bmatrix} \omega,_x \\ \omega,_y \\ \theta,_x \\ \theta,_y \end{bmatrix} = \begin{bmatrix} \omega,_s \\ \theta,_s \\ 0 \\ 0 \end{bmatrix}$$

Now investigate to find the two slopes of the line, $\frac{y,_s}{x,_s}$, that cause the determinant of the coefficient matrix to vanish. If the slopes are real and distinct the equations are called hyperbolic equations. It is shown below that this is the case. First, set the determinant of the coefficient matrix to zero.

$$\begin{vmatrix} x,_s & y,_s & 0 & 0 \\ 0 & 0 & x,_s & y,_s \\ 1 & 0 & \cos(2\cdot\theta) & \sin(2\cdot\theta) \\ 0 & 1 & \sin(2\cdot\theta) & -\cos(2\cdot\theta) \end{vmatrix} = x,_s{}^2\cdot\sin(2\cdot\theta) - 2\cdot x,_s\cdot y,_s\cdot\cos(2\cdot\theta) - y,_s{}^2\cdot\sin(2\cdot\theta) = 0$$

The roots are found as,

$$\frac{dy}{ds} = \frac{y,_s}{x,_s} = \frac{-\cos(2\cdot\theta)\pm 1}{\sin(2\cdot\theta)}$$

As the roots are real and distinct for real values of θ, the equations are hyperbolic equations. Let,

First shear direction, $\quad \dfrac{dy}{ds} = \dfrac{-\cos(2\cdot\theta)+1}{\sin(2\cdot\theta)}$

Second shear direction, $\quad \dfrac{dy}{ds} = \dfrac{-\cos(2\cdot\theta)-1}{\sin(2\cdot\theta)}$

N.B. – The slopes are perpendicular and the first shear direction is obtained by rotating the direction for the maximum principal stress by 45^o counter clockwise.

When a line in the x-y plane has one of the above slopes, the augmented matrix of the above matrix must have a rank of no more than two in order to avoid a contradiction. This is ensured for the first shear direction by setting the following determinant to zero.

$$\begin{vmatrix} x,_s & \dfrac{-\cos(2\cdot\theta)+1}{\sin(2\cdot\theta)}\cdot x,_s & 0 & \omega,_s \\ 0 & 0 & x,_s & \theta,_s \\ 1 & 0 & \cos(2\cdot\theta) & 0 \\ 0 & 1 & \sin(2\cdot\theta) & 0 \end{vmatrix} = 0$$

Expansion of this determinant yields,

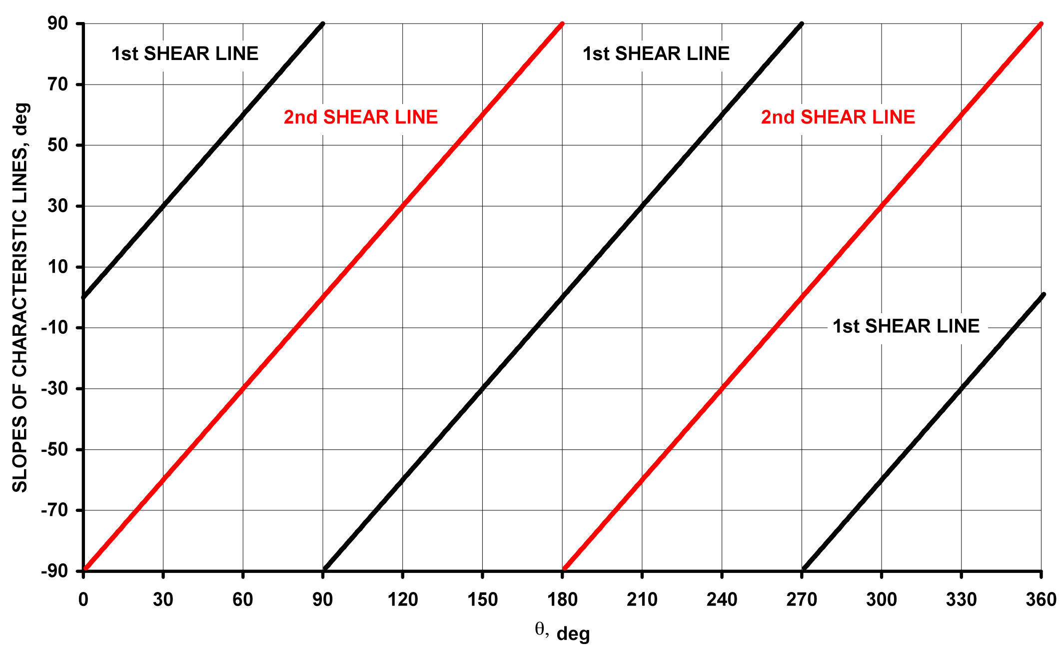
$$x,_s\cdot(-\theta,_s\cdot\cos(2\cdot\theta)) - \left(\frac{1-\cos(2\cdot\theta)}{\sin(2\cdot\theta)}\right)\cdot x,_s\cdot(\theta,_s\cdot\sin(2\cdot\theta)) + \omega,_s\cdot(x,_s) = x,_s\cdot(\omega,_s - \theta,_s) = 0$$

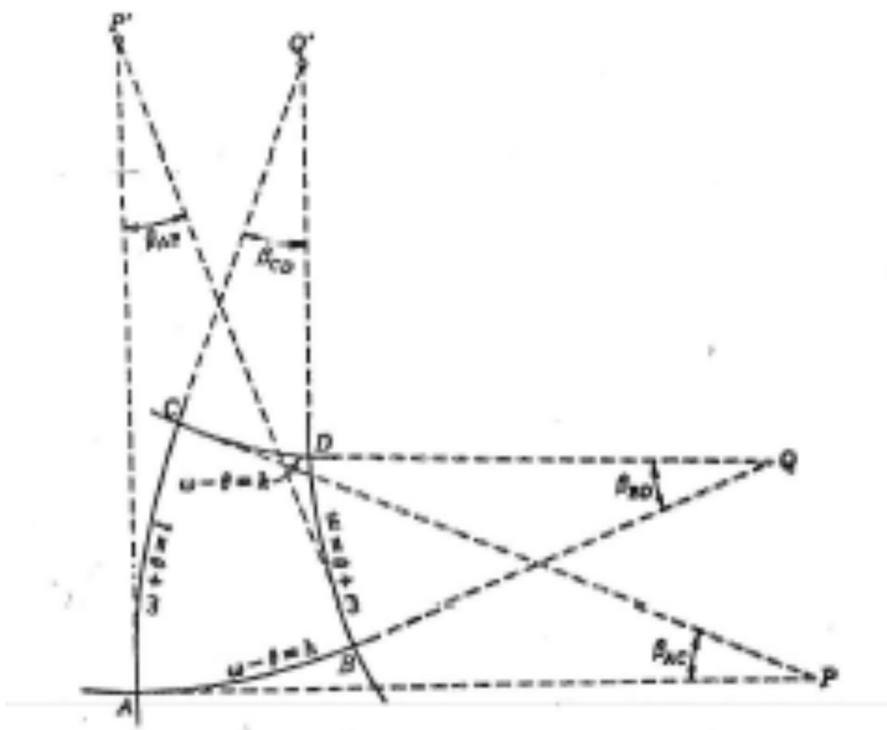
Therefore, $(\omega - \theta) =$ constant along a line whose tangent is parallel to the first shear direction. Such a line is called a first shear line.

For the second shear direction, $\dfrac{-\cos(2\cdot\theta)+1}{\sin(2\cdot\theta)}$ must be replaced by $\dfrac{-\cos(2\cdot\theta)-1}{\sin(2\cdot\theta)}$ in the above determinant. Expansion of the determinant yields $(\omega + \theta) =$ constant along a line whose tangent is parallel to the second shear direction. Such a line is called a second shear line.

The figure below shows the relationships between the slopes of the shear lines and θ. A family of shear lines refers to either a set of first shear lines or a set of second shear lines.

SLOPES OF CHARACTERISTIC LINES IN DEGREES VERSUS θ





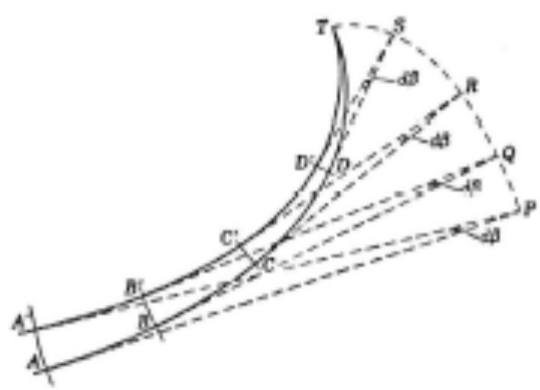
For the sketch above, the shear lines AB and CD are first shear lines while AC and BD are intersecting second shear lines. The conditions along shear lines derived above require that,

$$
\begin{aligned}
\omega_A - \theta_A &= \omega_B - \theta_B = h \\
\omega_C - \theta_C &= \omega_D - \theta_D = k \\
\omega_A + \theta_A &= \omega_C + \theta_C = l \\
\omega_B + \theta_B &= \omega_D + \theta_D = m
\end{aligned}
\quad \rightarrow \quad
\begin{aligned}
\theta_A &= \tfrac{1}{2}\cdot(l - h) \\
\theta_B &= \tfrac{1}{2}\cdot(m - h) \\
\theta_C &= \tfrac{1}{2}\cdot(l - k) \\
\theta_D &= \tfrac{1}{2}\cdot(m - k)
\end{aligned}
$$

Define,

$$
\begin{aligned}
\beta_{AC} &\equiv \theta_A - \theta_C = \tfrac{1}{2}\cdot(k - h) \\
\beta_{BD} &\equiv \theta_B - \theta_D = \tfrac{1}{2}\cdot(k - h)
\end{aligned}
$$

The fact that these two angles are equal is called Hencky's theorem: "The angle formed by the tangents of two fixed shear lines of one family at their points of intersection with a shear line of the second family does not depend on the choice of the intersecting shear line of the second family." A corollary to this theorem is that if a family of shear lines contains a straight line, it consists entirely of straight lines.



A second theorem, based on the above sketch is self evident and is called Prandtl's Theorem: "As we proceed along a fixed shear line of one family, the centers of curvature of the of the shear lines of the other family form an involute of the fixed shear line."

It is clear from the above figure that if ABCD and A'B'C'D' are first shear lines, point T is a point on the envelope of the first shear lines and the second shear line through T has a cusp there. The envelope of the shear lines of one family is the locus of the cusps of the shear lines of the second family. This means the envelope of the shear lines of one family is a limiting line across which the shear lines of the other family cannot be continued

Boundary Conditions in Plane Strain

A figure from P&H, in the formula sheet, shows the sign convention used everywhere except for the Mohr's circle construction. For this sign convention the stresses at a boundary point in the plastic region whose normal is inclined by the angle α, measured counter clockwise from the x axis, may be expressed in terms of ω and θ as follows,

$$N = 2\cdot k\cdot\omega + k\cdot\sin(2\cdot(\theta-\alpha))$$
$$T = -k\cdot\cos(2\cdot(\theta-\alpha))$$
$$N' = 2\cdot k\cdot\omega - k\cdot\sin(2\cdot(\theta-\alpha))$$

where N' is the interior normal stress at the boundary. When, as is usual, only T and N are specified on the boundary, the values of ω and θ are obtained as,

$$\theta = \alpha \;\; \pm\tfrac{1}{2}\cdot\left(\pi - \arccos\left(\frac{T}{k}\right)\right)$$
$$\omega = \tfrac{1}{2}\cdot\left(\frac{N}{k} - \sin(2\cdot(\theta-\alpha))\right)$$

This solution has two roots shown in the figure on the formula sheet. The value of N' is governed by the choice of sign in the equation for θ above

One must assume one of the solutions and proceed to solve the problem at hand. It is usually possible to "guess" the correct solution. In the long run, an incorrect choice will appear based on other boundary conditions not being satisfied and the correct choice can ultimately be found.

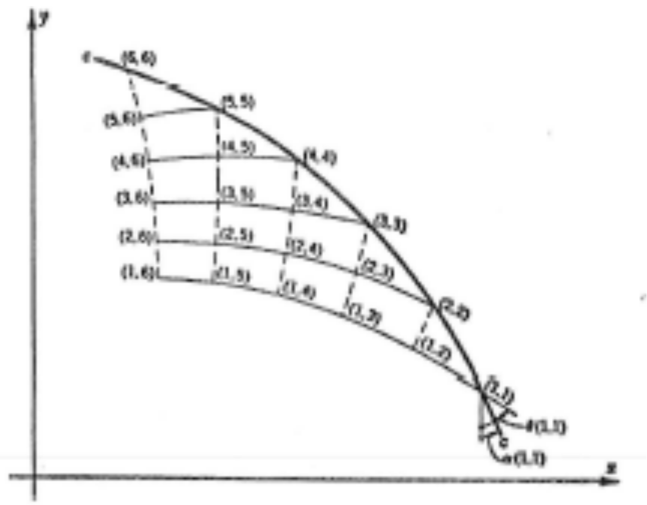
The following section discusses methods for solving for the stresses in the plastic region. In the section it is assumed that the proper choice for ω and θ on the boundary has been made.

Approximating Schemes for Determining Shear Lines in Plane Strain

An interesting property of hyperbolic equations is that specification of proper boundary conditions on part of the boundary determines the solution in part of the region. This region is called the domain of influence for that part of the boundary. This is very different from elasticity. The difference is because the elasticity equations are elliptic equations (no real lines for characteristics). There are three classical formulations for finding, in principle, the domain of influence. Each formulation has a different type of boundary condition.

First Boundary Condition Problem, the Cauchy Problem

ω and θ are specified along a line whose slope is nowhere equal to θ or $\theta + \frac{\pi}{2}$. The sketch below shows a case of this boundary value problem.



Arc c is the boundary and the light full lines and the dotted lines indicate first and second shear lines, respectively. The values of ω and θ are specified at points (1,1) … (6,6). The objective is to find the shape of the domain of influence with corners at (1,1), (6,6) and (1,6) as well as the values of ω and θ in the interior at all the indicated points. Consider point (1,2), the conditions along the first and second shear lines through this point yield,

$$\omega(1,2) - \theta(1,2) = \omega(1,1) - \theta(1,1)$$
$$\omega(1,2) + \theta(1,2) = \omega(2,2) + \theta(2,2)$$

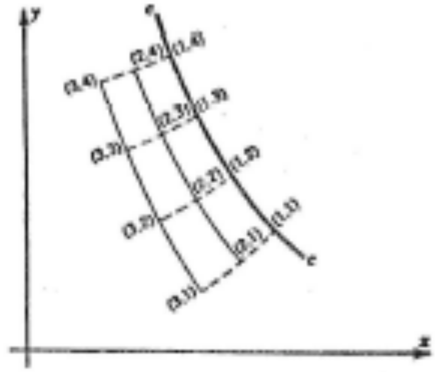
These equations may be solved to give,

$$\omega(1,2) = \tfrac{1}{2} \cdot (\omega(2,2) + \omega(1,1) + \theta(2,2) - \theta(1,1))$$
$$\theta(1,2) = \tfrac{1}{2} \cdot (\omega(2,2) - \omega(1,1) + \theta(2,2) + \theta(1,1))$$

All the terms on the right-hand side of the two equations above are known so $\omega(1,2)$ and $\theta(1,2)$ are determined. The physical location of (1,2) can be approximated by the intersection of the first shear line through (1,1) and the second shear line through (2,2). The same procedure can be followed for points (2,3), (3,4), (4,5) and (5,6). With the values of ω and θ known and the location of the points approximated. The same procedure can be repeated for points (1,3), (2,4), (3,5) and (4,6). Eventually point (1,6) is reached and the problem is solved. P&H present a refinement for the point location calculation but these days no one works these problems graphically.

Second Boundary Condition Problem, the Riemann Problem

ω and θ are specified along a line that is a first shear line. . In addition, an intersecting second shear line is defined along which θ is known. The sketch below illustrates this boundary condition problem. ω and θ are known for points (1,1) … (1,4) and θ is known for points (2,1) and (3,1). The domain of influence is bounded by points (1,1), (1,4), (3,4) and (3,1).



Consider point (2,2)and apply Hencky's theorem to find,

$$\theta(2{,}2) - \theta(1{,}2) = \theta(2{,}1) - \theta(1{,}1)$$

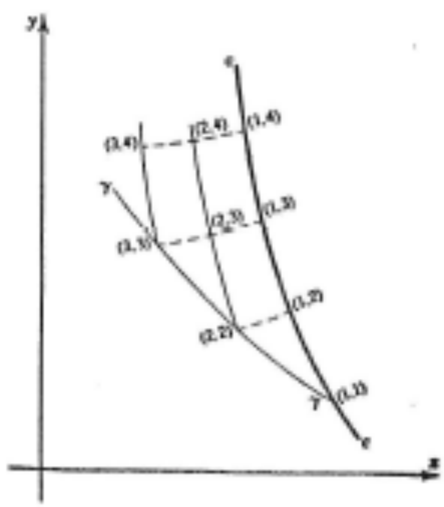
so that,

$$\theta(2{,}2) = \theta(1{,}2) + \theta(2{,}1) - \theta(1{,}1)$$

Since all terms on the right-hand side are known $\theta(2{,}2)$ is determined. The location of point (2,2) is approximated from the intersection of the first shear line direction through (2,1) and the second shear line direction through (1,2) The value of ω is found from the condition that $\omega + \theta$ is constant on the second shear line through point (1,2). This procedure can be continued through the entire domain of influence.

Third Boundary Condition Problem, the Mixed Problem

Specify a first shear, c-c in the figure below, and another line γ–γ, not along a shear line, along which ω or θ is specified.



If, for example, ω is specified on γ–γ, then,

$\omega(1,2) + \theta(1,2) = \omega(2,2) + \theta(2,2)$

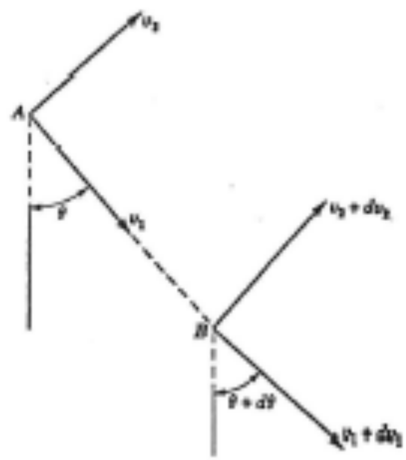
and then,

$\theta(2,2) = \omega(1,2) - \omega(2,2) + \theta(1,2)$

This sort of scheme can be used to extend the solution through the domain of influence.

Velocity Fields in Plane Strain

On the shear lines there is no axial strain rate. The figure below shows two points, A and B, along a first shear line.



The condition that the extension rate vanish along this first shear line is,

$v_{1,s} - v_2 \cdot \theta_{,s} = 0$

and for a second shear line,

$v_{2,s} + v_1 \cdot \theta_{,s} = 0$

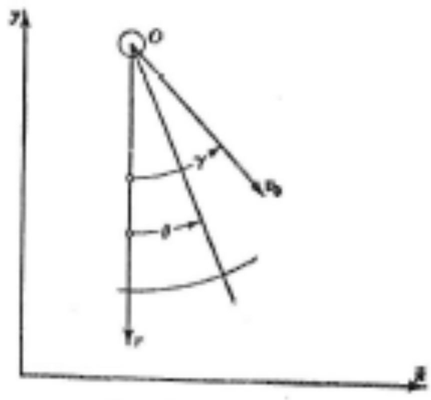
1. The velocity along a shear line is constant if,
   A, It is straight, $\theta_{,s} = 0$ along line.
   B. It is a streamline, $v_{\text{other family}} = 0$
2. P&H give a geometric scheme for finding velocities that is obvious.
3. A rigid-plastic boundary must be a shear line or a limiting line.
4. On a rigid-plastic boundary v_n must be continuous, therefore = 0. If v_t is continuous then the resolved velocity is zero. Therefore, v_t is usually discontinuous across a rigid-plastic boundary

Families of Straight Shear Lines in Plane Strain

An apparently trivial case is for the first and second shear lines to form a rectangular grid aligned with the x and y axes. In this case the velocity field is,

$v_x = f(y)$, $v_y = g(x)$ where both f(y) and g(x) are arbitrary

When only one shear line family is straight, the shear line pattern is called a fan. The simplest case is the centered fan shown below. The straight shear lines are taken to be first shear lines so that the second shear lines are concentric circles centered at point O.



On each of the first shear lines θ is constant and, since $\omega - \theta$ is also constant, ω is constant.

For the second shear lines, $\omega + \theta$ = constant = c in the entire fan.

ω is independent of r as are σ_x, σ_y, τ.

The stresses in this centered fan are,

$$\sigma_r = \sigma_\theta = 2 \cdot k \cdot \omega = 2 \cdot k \cdot (c - \theta), \quad \tau_{r\theta} = k$$

For the velocity components $v_1 = v_r$ and $v_2 = v_\theta$. The inextensibility conditions require that,

$v_r = -h'(\theta)$, $v_q = h(\theta) + g(r)$

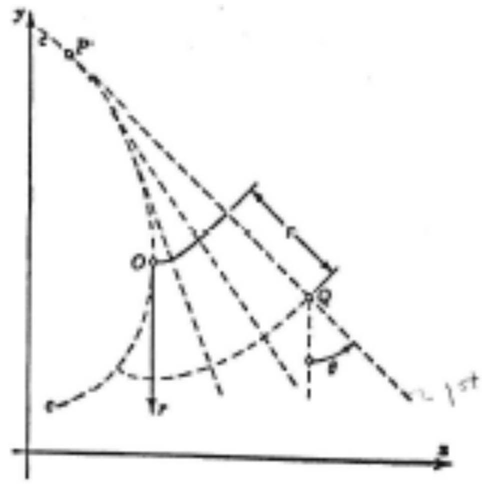
If there is a plastic region at point O and the velocity at point O is v_O directed as indicated in the above figure, the functions h(θ) and g(r) are determined as,

$$h(\theta) = v_O \cdot \sin(\gamma - \theta), \quad g(0) = 0$$

and then,

$$v_r = v_O \cdot \cos(\gamma - \theta), \quad v_\theta = v_O \cdot \sin(\gamma - \theta) + g(r) \quad g(0) = 0$$

A case of a non-centered fan is shown in the figure below.



The first shear lines are all tangent to a base curve while the second shear lines are involutes of the base curve. Like the centered fan, $\omega + \theta = c$ in this fan. The "origin", point O, is chosen where θ is equal to zero and the coordinate r to point Q is measured from the involute going through O. With this coordinate system the solution for the stresses is the same as for the centered fan and the velocities are given by,

$v_r = -h'(\theta)$, $v_q = h(\theta) + g(r)$

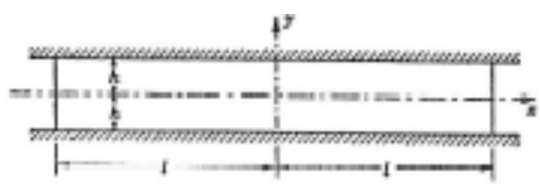
Before giving an example of a straight limiting line, some characteristics of limiting lines are reviewed here. The envelope of one family of shear lines is the locus of the cusps of the shear lines of the other family. Consider a limiting line with normal stress N, shear stress T and interior normal stress N'. P&H discuss the nature of the derivatives of the stresses and velocity with the result that,

$$\frac{\partial N}{\partial n} = 0 \qquad \frac{\partial T}{\partial n} = \text{finite} \qquad \left|\frac{\partial N'}{\partial n}\right| = \text{infinite}$$

$$\frac{\partial N}{\partial s} = \text{finite} \qquad \frac{\partial T}{\partial s} = 0 \qquad \frac{\partial N'}{\partial s} = \text{finite}$$

$$\left|\frac{\partial v_s}{\partial n}\right| = \text{infinite}$$

The problem described in the figure below is due to Prandtl and contains a straight limiting line.



The solution given by Prandtl is not valid near x = 0 or at the ends $x = \pm l$ so it is best to think of the case where $l \gg h$. In the "valid" regions,

$$\sigma_x = -p + k\cdot\left(\frac{x}{h} + 2\cdot\sqrt{1-\frac{y^2}{h^2}}\right)$$

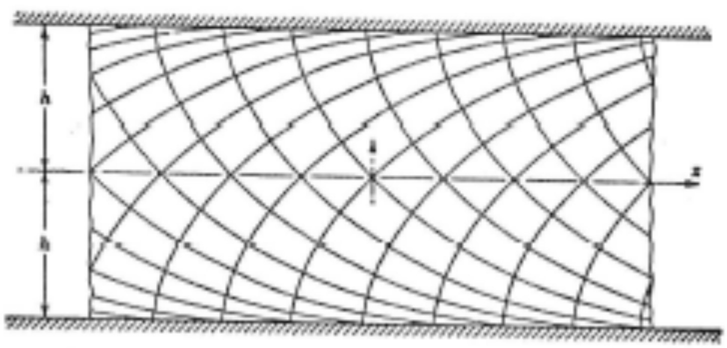
$$\sigma_y = -p + k\cdot\frac{x}{h}$$

$$\tau = -k\cdot\frac{y}{h}$$

$$v_x = c\cdot\left(\frac{x}{h} + 2\cdot\sqrt{1-\frac{y^2}{h^2}}\right)$$

$$v_y = -c\cdot\frac{y}{h}$$

The shear line pattern in this region is shown below.

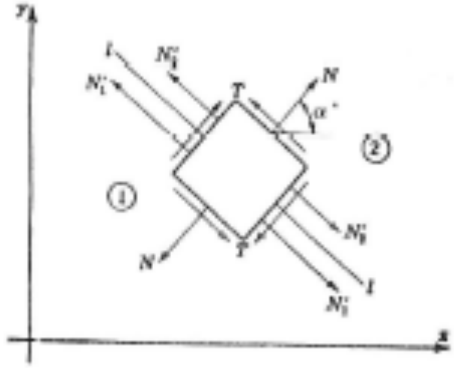


The shear lines are cycloids.

Lines of Discontinuity in Plane Strain

Bending of a beam composed of a Mises material will develop a stress discontinuity that is familiar to mechanical and structural engineers. In this case it is clear that the discontinuity is the remains of an elastic region and should be regarded as an elastic, inextensible filament. P&H show that this is true for all stress discontinuities in plane strain.

A formal approach to this plane strain problem can be based on the figure below where the different sides of the discontinuity are numbered 1 and 2.



The conditions that the stresses normal (N) and tangential (T) to the stress discontinuity are continuous lead to (see formula sheet, Appendix 4),

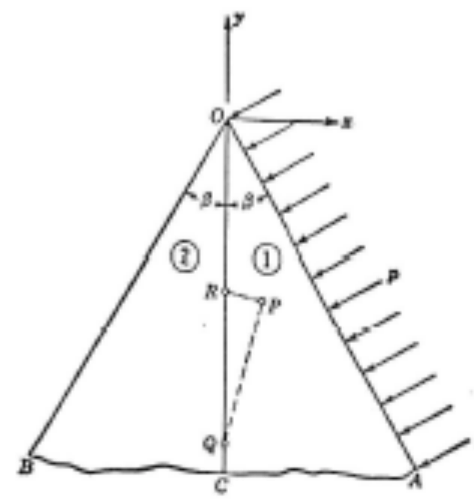
$$2\cdot\omega_1 + \sin(2\cdot(\theta_1 - \alpha)) = 2\cdot\omega_2 + \sin(2\cdot(\theta_2 - \alpha))$$
$$\cos(2\cdot(\theta_1 - \alpha)) = \cos(2\cdot(\theta_2 - \alpha))$$

Prager has determined that the roots of interest here are given by,

$$\omega_2 = \omega_1 + \sin(2\cdot(\theta_1 - \alpha))$$
$$2\cdot(\theta_2 - \alpha) = -2\cdot(\theta_1 - \alpha) \pm 2\cdot n\cdot\pi\,, \quad \text{n is an integer}$$

These conditions are called the Prager jump conditions. Note that $\alpha = \frac{1}{2}\cdot(\theta_1 + \theta_2) \mp n\cdot\pi$ so that the line of discontinuity bisects the corresponding shear lines on each side of the discontinuity.

The sketch below is for a problem where a stress field may be found using a stress discontinuity.



The wedge is loaded by the uniform pressure, p, on the right flank. Referring to the formula sheet,

$$\text{On OA}: \alpha = \beta\,, \quad N = -p\,, \quad T = 0$$
$$\text{On OB}: \alpha = \pi - \beta\,, \quad N = 0\,, \quad T = 0$$

yields,

$$\text{On OA}: \theta = \beta \pm \tfrac{\pi}{4}\,, \quad \omega = \tfrac{1}{2}\cdot\left(-\frac{p}{k} \mp 1\right)$$
$$\text{On OB}: \theta = \pi - \beta \pm \tfrac{\pi}{4}\,, \quad \omega = \mp\tfrac{1}{2}$$

Although it is possible to proceed on a mathematical basis, like P&H, it is pretty clear that on the left flank the interior normal stress is compressive. The maximum normal stress on the left flank is zero so the value of θ is $\left(\frac{5}{4}\cdot\pi - \beta\right)$ and the upper sign must be taken for the above equation for OB. Since the discontinuity, OC, must bisect the first shear lines, the lower sign must be chosen for the above equation for OA. Consequently,

$$\theta_1 = \beta - \tfrac{\pi}{4}, \quad \omega_1 = \tfrac{1}{2}\cdot\left(-\frac{p}{k}+1\right) \quad \text{and} \quad \theta_2 = \tfrac{5}{4}\cdot\pi - \beta, \quad \omega_2 = -\tfrac{1}{2}$$

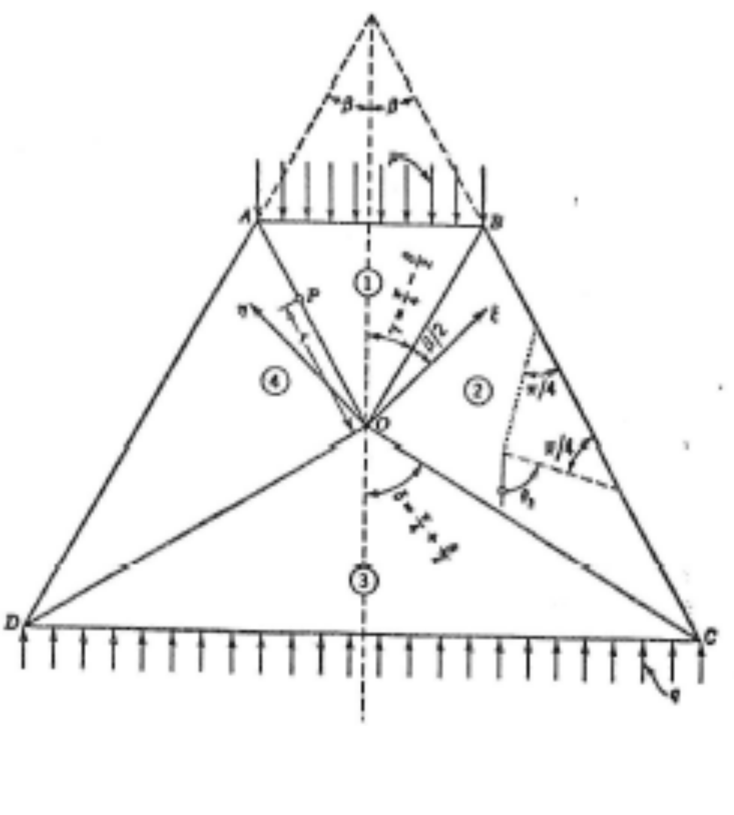
and the first jump condition then gives,

$$p = 2\cdot k\cdot(1 - \cos(2\cdot\beta))$$

For the velocity field, recall that OC must be treated as an inextensible filament. It also makes sense to set $v_y = 0$ on this filament. It turns out that v_x may be an arbitrary continuous function. It is clear that the velocity at any point P at the intersection of the shear lines from R and Q can be found from the inextensibility of the shear lines.

P&H go deeper into the discontinuity study but it won't be pursued here except to mention an obvious requirement of the solutions with stress discontinuities. The work done on the body by the presence of a discontinuity must be positive.

One solution with stress discontinuities is shown below because it will be compared with another solution later.

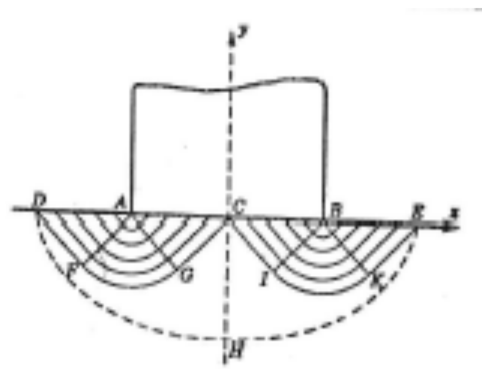


The top pressure, p, for incipient plastic flow is given in P&H as,

$$p = 2\cdot k\cdot(1 + \sin(\beta))$$

P&H CHAPTER 6, PLANE STRAIN, SPECIFIC PROBLEMS

Punch Problem



Find θ by rotating the σ_{max} *face* by $+\frac{\pi}{4}$

In DFA:

$\sigma_x = -2k$ AF is a 1st shear line $\omega_{DFA} = -\frac{1}{2}$

$\sigma_y = 0$ DF is a 2nd shear line $\theta_{DFA} = -\frac{\pi}{4}$

$\tau = 0$

AG must be a first shear line and

$$\theta_{AGC} = \theta_{DFA} + \frac{\pi}{2} = +\frac{\pi}{4}$$

Using the condition that $\omega + \theta$ is constant on a second shear line,

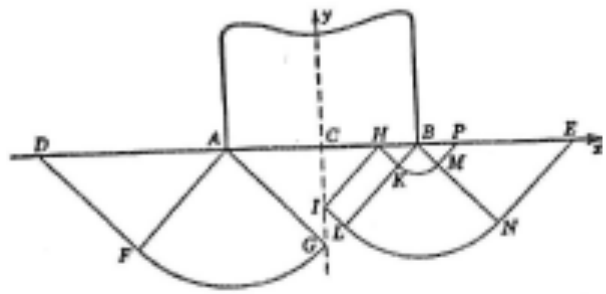
$$\omega_{DFA} + \theta_{DFA} = \omega_{AGC} + \theta_{AGC} \quad \rightarrow \quad \omega_{AGC} = -\frac{1}{2}\cdot(1+\pi)$$

In AGC:

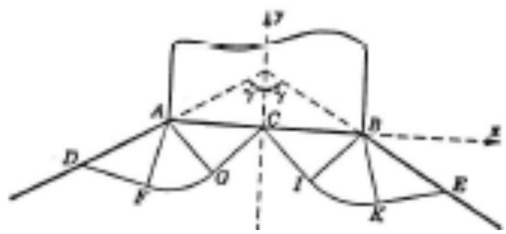
$$\begin{aligned} \sigma_x &= 2\cdot k\cdot\omega_{AGC} + k\cdot\sin(2\cdot\theta_{AGC}) & & \sigma_x = -\pi\cdot k \\ \sigma_y &= 2\cdot k\cdot\omega_{AGC} - k\cdot\sin(2\cdot\theta_{AGC}) & \rightarrow \quad & \sigma_y = -(2+\pi)\cdot k \\ \tau &= -k\cdot\cos(2\cdot\theta_{AGC}) & & \tau = 0 \end{aligned}$$

And the uniform pressure on AB is $(2+\pi)\cdot k$

Two alternative shear line solutions are shown below. The computed punch force is the same as for the above set of shear lines.



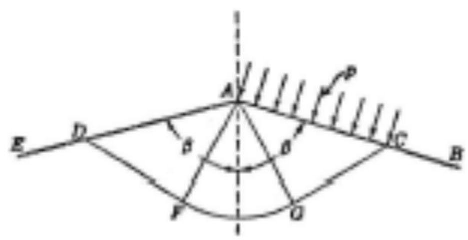
The sketch below is for a variation on the punch problem. A uniform pressure, p, is applied on AB.



Obviously, the solution for the pressure is obtained from the punch problem by replacing $\frac{\pi}{2}$ by γ so that,

$$p = 2 \cdot k \cdot (1 + \gamma)$$

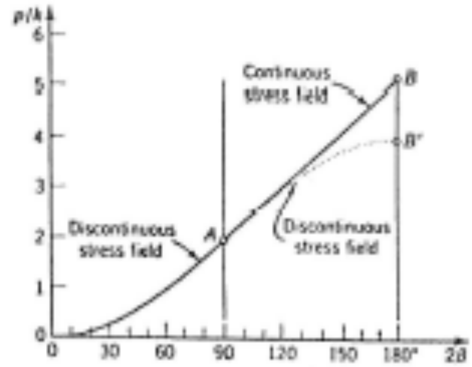
Lateral loading on part of the flank of an obtuse wedge is shown in the sketch below.



The pressure for incipient plastic flow is found by replacing γ in the preceding solution by $2 \cdot \beta - \frac{\pi}{2}$ to obtain,

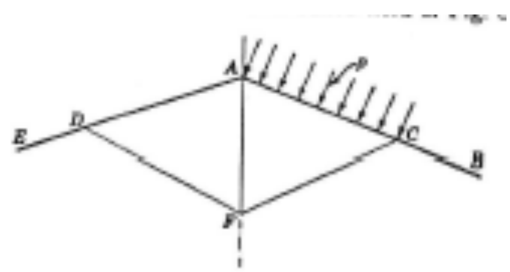
$$p = 2 \cdot k \cdot \left(1 + 2 \cdot \beta - \frac{\pi}{2}\right)$$

Earlier a solution for this problem was presented in connection with studying stress discontinuities. P&H give the figure below, their Figure 62, comparing the two solutions.

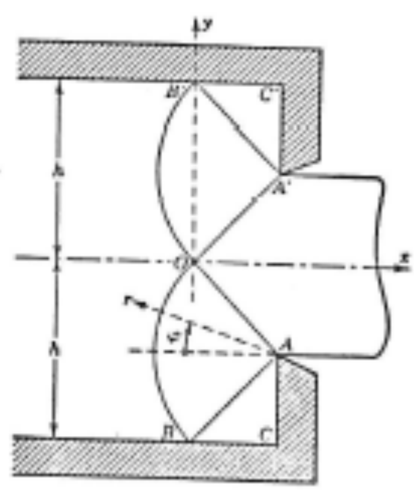


An extremely important conclusion can be drawn from this comparison. The point is made by the following quote from P&H.

“As is seen from Fig. 62, the pressure value corresponding to the discontinuous stress field for the obtuse wedge is smaller than that corresponding to the continuous solution. As the pressure is gradually increased, starting from zero, one might therefore expect that plastic flow sets in under the smaller pressure corresponding to the discontinuous stress pattern before the pressure has reached the higher value required by the continuous stress pattern. This is not the case, however, because there is no field of plastic flow associated with the discontinuous stress field.”



A frictionless extrusion problem for reducing the thickness of a slab by 50% in steady flow is shown below. Many forming and machining operations may be analyzed as steady flow problems.



$\sigma_x = \tau = 0$ on OA & OA'

$\therefore \sigma_y = -2 \cdot k$ on OA & OA' and $\omega = -\frac{1}{2}$ $\theta = \frac{\pi}{4}$

OA is 1st shear line

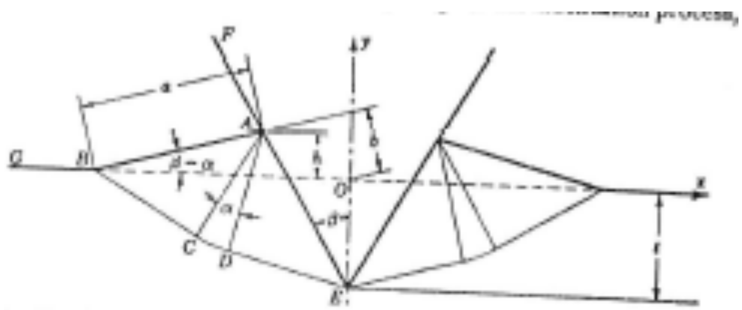
on AB $\omega = -\frac{1}{2} \cdot (1 + \pi)$ $\theta = \frac{3 \cdot \pi}{4}$ $\sigma_x = -(2 + \pi) \cdot k$ $\sigma_y = -\pi \cdot k$ $\tau = 0$

The extrusion pressure, p, is then given by,

$$p = -\left(1 + \frac{\pi}{2}\right) \cdot k$$

and P&H give a corresponding velocity field.

A frictionless wedge indentation problem is shown in the figure below.



For this problem, as the wedge indents the surface, the configuration is simply magnified. P&H call this kind of problem pseudo-steady plastic flow.

AE = AB = a

$$h = a \cdot \sin(\beta - \alpha)$$

$$t = a \cdot \cos(\beta) - h = a \cdot \cos(\beta) - a \cdot \sin(\beta - \alpha) \quad \rightarrow \quad a = \frac{t}{\cos(\beta) - \sin(\beta - \alpha)}$$

$$h = a \cdot \sin(\beta - \alpha) = \frac{t \cdot \sin(\beta - \alpha)}{\cos(\beta) - \sin(\beta - \alpha)}$$

$$b = (a \cdot \cos(\beta - \alpha) + a \cdot \sin(\beta)) \cdot \sin(\beta - \alpha) \quad \rightarrow \quad b = \frac{t \cdot \sin(\beta - \alpha) \cdot (\cos(\beta - \alpha) + \sin(\beta))}{\cos(\beta) - \sin(\beta - \alpha)}$$

$$v_1 = 0, \quad v_2 \cdot \sin\left(\frac{\pi}{4}\right) = 1 \cdot \sin(\beta) \quad \rightarrow \quad v_2 = \sqrt{2} \cdot \sin(\beta)$$

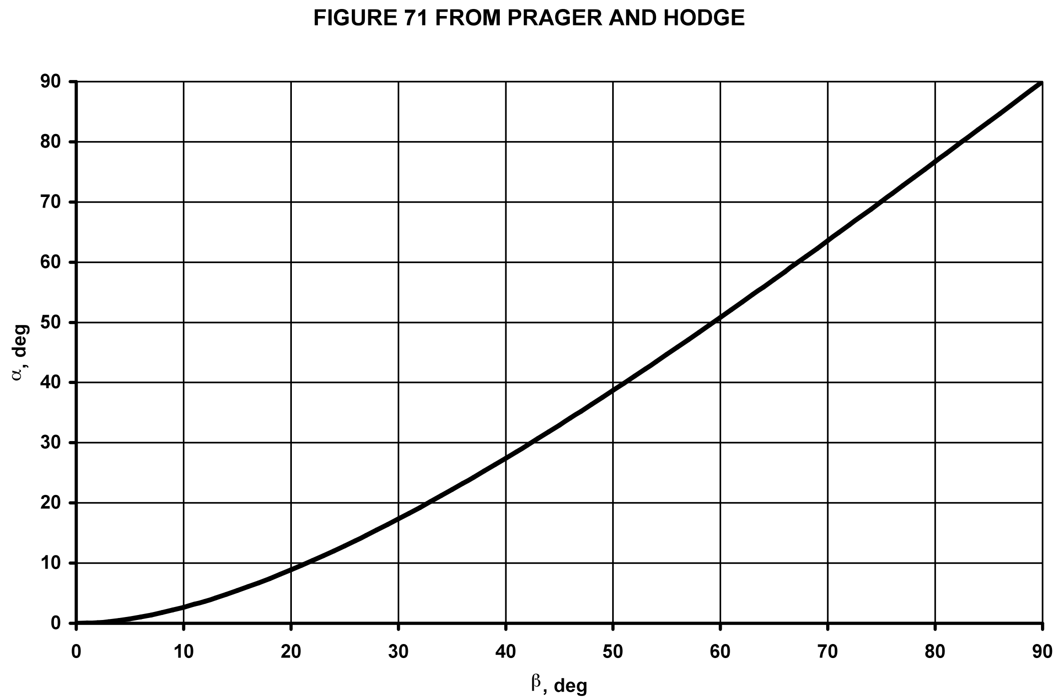
The incompressibility condition is,

$$\frac{1}{2} \cdot h \cdot (a \cdot \cos(\beta - \alpha) + h \cdot \tan(\beta)) = \frac{1}{2} \cdot t^2 \cdot \tan(\beta)$$

This may be reduced to,

$$\sin(\beta - \alpha) \cdot (\cos(\beta - \alpha) + \sin(\beta - \alpha) \cdot \tan(\beta)) = \tan(\beta) \cdot (\cos(\beta) - \sin(\beta - \alpha))^2$$

This equation was solved numerically and the resulting plot is given below,



P&H give the velocity field for this problem.

P&H CHAPTER 7, PLANE STRAIN: PRINCIPLE OF VIRTUAL WORK

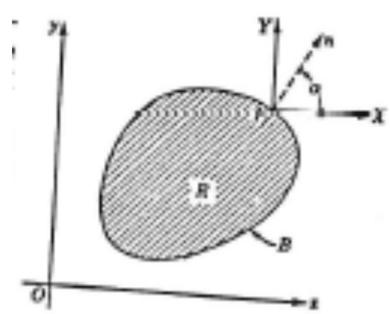
Limit analysis is a mathematical method used to find upper and lower bounds on the limiting, plastic load state.

Proportional loading is assumed. That is, all loads are proportional to a single monotonically increasing parameter.

The Principle of Virtual Work plays a large role in the limit analysis theorems. It is described below and subsequently proved.

Principle of Virtual Work:

1. Assume σ_{ij} and v_i are continuously differentiable with respect to x and y.
2. No body forces.
3. σ_{ij} satisfies equilibrium conditions, $\sigma_{ij,j} = 0$.
4. The velocities are used to determine the strain rates
   $\dot{\varepsilon}_x = v_{x,x}$, $\dot{\varepsilon}_y = v_{y,y}$, $\dot{\gamma} = v_{x,y} + v_{y,x}$
5. Boundary loads equilibrate with stresses at boundary
   $X = \sigma_x \cdot \cos(\alpha) + \tau \cdot \sin(\alpha)$
   $Y = \sigma_y \cdot \sin(\alpha) + \tau \cdot \cos(\alpha)$



Then,

$$\int_R \left(\sigma_x \cdot \dot{\varepsilon}_x + \sigma_y \cdot \dot{\varepsilon}_y + \tau \cdot \dot{\gamma}\right) dA = \oint_B \left(X \cdot v_x + Y \cdot v_y\right) dS$$

or the rate of work absorbed by the body equals the rate of external work done on the body.

Proof of theorem:

$$\int_R \left(\sigma_x \cdot \dot{\varepsilon}_x + \sigma_y \cdot \dot{\varepsilon}_y + \tau \cdot \dot{\gamma}\right) dA = \int_R \left(\sigma_x \cdot v_{x,x} + \sigma_y \cdot v_{y,y} + \tau \cdot v_{x,y} + \tau \cdot v_{y,x}\right) dA$$
$$= \int_R \left[\left(\sigma_x \cdot v_x + \tau \cdot v_y\right)_{,x} + \left(\sigma_y \cdot v_y + \tau \cdot v_x\right)_{,y} - v_x \cdot \left(\sigma_{x,x} + \tau_{,y}\right) - v_y \cdot \left(\sigma_{y,y} + \tau_{,x}\right)\right] dA$$
$$= \oint_B \left[\left(\sigma_x \cdot v_x + \tau \cdot v_y\right) n_x + \left(\sigma_y \cdot v_y + \tau \cdot v_x\right) n_y\right] dB$$
$$= \oint_B \left[\left(\sigma_x \cdot \cos(\alpha) + \tau \cdot \sin(\alpha)\right) \cdot v_x + \left(\sigma_y \cdot \sin(\alpha) + \tau \cdot \cos(\alpha)\right) v_y\right] dB$$
$$= \oint_B \left(X \cdot v_x + Y \cdot v_y\right) dS$$

The theorem is also valid for rates of the stresses so long as the equilibrium equations are satisfied in time. That is,

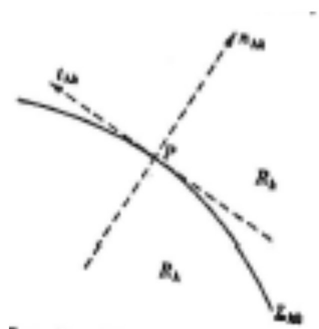
$$\int_R \left(\dot{\sigma}_x \cdot \dot{\varepsilon}_x + \dot{\sigma}_y \cdot \dot{\varepsilon}_y + \dot{\tau} \cdot \dot{\gamma}\right) dA = \oint_B \left(\dot{X} \cdot v_x + \dot{Y} \cdot v_y\right) dS$$

Influence of discontinuities on Principle of Virtual Work:

Stress discontinuities:

Across a discontinuity the normal and shear stress components are continuous so, if v is continuous, there is no contribution to the Principle of Virtual Work.

Velocity discontinuities:



n_{hk} is directed from sub region R_h into sub region R_k. t_{hk} is obtained from n_{hk} with a 90^o counter clockwise rotation. n_{hk} and t_{hk} are unit length vectors and,

$$v_n^{(k)} = v_n^{(h)}$$
$$v_t^{(k)} \neq v_t^{(h)}$$

$N^{(hk)}$ and $T^{(hk)}$ are positive if directed like n_{hk} and t_{hk}

The contribution to the work is $T^{(hk)} \cdot \left(v_t^{(k)} - v_t^{(k)}\right) dl_{hk}$

Principle of Virtual Work with stress and velocity discontinuities:

$$\int_R \left(\sigma_x \cdot \dot{\varepsilon}_x + \sigma_y \cdot \dot{\varepsilon}_y + \tau \cdot \dot{\gamma}\right) dA = \oint_B \left(X \cdot v_x + Y \cdot v_y\right) dS + \sum \int_{L_{hk}} T^{(hk)} \cdot \left(v_t^{(h)} - v_t^{(k)}\right) dl_{hk}$$

and

$$\int_R \left(\dot{\sigma}_x \cdot \dot{\varepsilon}_x + \dot{\sigma}_y \cdot \dot{\varepsilon}_y + \dot{\tau} \cdot \dot{\gamma}\right) dA = \oint_B \left(\dot{X} \cdot v_x + \dot{Y} \cdot v_y\right) dS + \sum \int_{L_{hk}} \dot{T}^{(hk)} \cdot \left(v_t^{(h)} - v_t^{(k)}\right) dl_{hk}$$

Plane Strain, Limit Analysis Using Prandtl-Reuss, Theorems of Greenberg, Drucker and Prager

First boundary condition (BC#1): Arc along which surface tractions are specified

Second boundary condition (BC#2) Arc along which displacement vector must *vanish*

To have a nontrivial problem part or all of boundary must be BC#1.

Proportional loading

Contained flow: linearized strains and displacements are acceptable

Looking for the load that causes impending plastic flow. At impending plastic flow, plastic strain increases under constant surface tractions for the first time.

$$S = \text{safety factor} \equiv \frac{(\text{surface traction at impending plastic flow})}{(\text{reference surface traction})}$$

where the reference surface traction is less than surface traction at impending plastic flow (S > 1).

Define a statically admissible stress field as one which,

1. Satisfies the force equilibrium equations
2. Satisfies BC#1
3. Satisfies the yield inequality, $\left(\sigma_x - \sigma_y\right)^2 + 4 \cdot \tau^2 - 4 \cdot k^2 \le 0$, everywhere

Obviously, the actual impending plastic flow solution satisfies the statically admissible stress field conditions. Let highest multiplier of the reference surface traction that is statically admissible be m_s.

First limit analysis theorem:

$m_s \le S$

Define a kinematically admissible velocity field as one which,

1. The incompressibility condition, $v_{i,I} = 0$
2. Satisfies BC#2
3. $\oint_B \left(X \cdot v_x + Y \cdot v_y\right) dS > 0$

Obviously, the actual impending plastic flow solution satisfies the kinematically admissible velocity conditions. For this kinematically admissible velocity field compute,

$$m_k = k \cdot \frac{\int_R \dot{\Gamma} \cdot dA + \sum \int_{L_{hk}} \left| v_t^{(k)} - v_t^{(h)} \right| \cdot dl_{hk}}{\oint_B \left(X \cdot v_x + Y \cdot v_y \right) dS}$$

where

$$\dot{\Gamma} = \sqrt{\left(v_{x,x} - v_{y,y}\right)^2 + \left(v_{x,y} + v_{y,x}\right)^2} = \text{maximum shear rate}$$

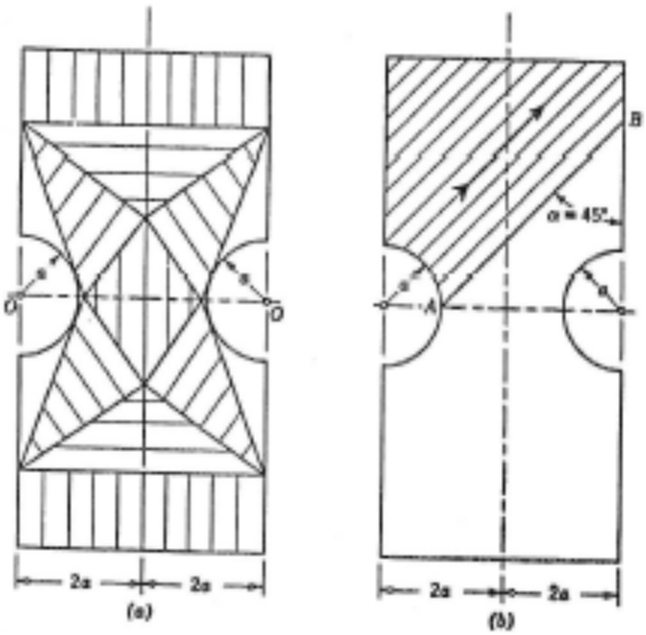
Second limit analysis theorem:

$$S \le m_k$$

so that,

$$m_s \le S \le m_k$$

An illustrative problem applying the above two theorems is shown below



From the shear line field having four stress discontinuities,

$$m_s = \frac{1.26 \cdot k}{T}$$

From the shear pattern on the right side of the above figure,

$$m_k = \frac{k \cdot \left(3 \cdot a \cdot \sqrt{2}\right)}{\left(\frac{4 \cdot a \cdot T}{\sqrt{2}}\right)} = \frac{1.5 \cdot k}{T}$$

so that,

$$\frac{1.26 \cdot k}{T} \le S \le \frac{1.50 \cdot k}{T}$$

The above two limit theorems are proved below. This is accomplished in two steps. The first step proves three auxiliary theorems and the second step uses these theorems to prove the limit analysis theorems.

First auxiliary theorem:

During incipient plastic flow $\dot{\sigma}_{ij} = 0$ in the whole body.

Second auxiliary theorem:

Let $\sigma_x{}^o, \sigma_y{}^o, \tau^o$ be a statically admissible stress field while σ_x, σ_y, τ and $\dot{\varepsilon}_x'', \dot{\varepsilon}_y'', \dot{\gamma}''$ are the actual stresses and plastic strain rates at incipient flow. Then,

$$\left(\sigma_x - \sigma_x{}^o\right)\dot{\varepsilon}_x'' + \left(\sigma_y - \sigma_y{}^o\right)\dot{\varepsilon}_y'' + \left(\tau - \tau^o\right)\dot{\gamma}'' \ge 0$$

Third auxiliary theorem:

Let $\dot{\varepsilon}_x{}^*, \dot{\varepsilon}_y{}^*, \dot{\gamma}^*$ denote the strain rates from a kinematically admissible velocity field $v_x{}^*$, $v_y{}^*$ and σ_x, σ_y, τ the actual stresses. Then,

$$\sigma_x \cdot \dot{\varepsilon}_x{}^* + \sigma_y \cdot \dot{\varepsilon}_y{}^* + \tau \cdot \dot{\gamma}^* \le k \cdot \dot{\Gamma}^* \quad \text{where} \quad \dot{\Gamma}^* = \sqrt{\left(\dot{\varepsilon}_x{}^* - \dot{\varepsilon}_y{}^*\right)^2 + \dot{\gamma}^{*2}}$$

Proof of first auxiliary theorem:

Assume stress rates don't vanish while $\dot{X}$ and $\dot{Y}$ vanish and find a contradiction.

In this proof let the velocities, strain rates and stresses be the actual values at incipient plastic flow.

Apply virtual work theorem,

$$\int_R \left(\dot{\sigma}_x \cdot \dot{\varepsilon}_x + \dot{\sigma}_y \cdot \dot{\varepsilon}_y + \dot{\tau} \cdot \dot{\gamma}\right) dA = \oint_B \cancelto{0}{\left(\dot{X} \cdot v_x + \dot{Y} \cdot v_y\right)} dS + \sum \int_{L_{hk}} \cancelto{0}{\dot{T}^{(hk)}} \cdot \left(v_t{}^{(h)} - v_t{}^{(k)}\right) dl_{hk}$$

so,

$$\int_R \left(\dot{\sigma}_x \cdot \dot{\varepsilon}_x + \dot{\sigma}_y \cdot \dot{\varepsilon}_y + \dot{\tau} \cdot \dot{\gamma}\right) dA = \int_R \left[\tfrac{1}{2} \cdot \left(\dot{\sigma}_x - \dot{\sigma}_y\right) \left(\dot{\varepsilon}_x - \dot{\varepsilon}_y\right) + \dot{\tau} \cdot \dot{\gamma}\right] dA$$

$$= \int_R \left[\tfrac{1}{2} \cdot \left(\dot{\sigma}_x - \dot{\sigma}_y\right) \left(\dot{\varepsilon}_x{}' - \dot{\varepsilon}_y{}'\right) + \dot{\tau} \cdot \dot{\gamma}'\right] dA + \int_R \left[\tfrac{1}{2} \cdot \left(\dot{\sigma}_x - \dot{\sigma}_y\right) \left(\dot{\varepsilon}_x{}'' - \dot{\varepsilon}_y{}''\right) + \dot{\tau} \cdot \dot{\gamma}''\right] dA$$

$$= \int_R \frac{1}{4 \cdot G} \cdot \left[\left(\dot{s}_x - \dot{s}_y\right)^2 + 4 \cdot \dot{\tau}^2\right] dA + \int_R \frac{\lambda}{4 \cdot G} \cdot \left[\left(s_x - s_y\right)\left(\dot{s}_x - \dot{s}_y\right) + 4 \cdot \tau \cdot \dot{\tau}\right] dA$$

$$= \int_R \frac{1}{4 \cdot G} \cdot \left[\left(\dot{s}_x - \dot{s}_y\right)^2 + 4 \cdot \dot{\tau}^2\right] dA + \int_R \frac{\lambda}{8 \cdot G} \cdot \frac{d}{dt}\left[\left(s_x - s_y\right)^2 + 4 \cdot \tau^2\right] dA$$

$$= \int_R \frac{1}{4 \cdot G} \cdot \left[\left(\dot{s}_x - \dot{s}_y\right)^2 + 4 \cdot \dot{\tau}^2\right] dA + \int_R \frac{\lambda}{8 \cdot G} \cdot \cancelto{0}{\frac{d}{dt}\left[k^2\right]} dA$$

This integral must vanish in order to satisfy the virtual work condition. This requires that,

$$\dot{\sigma}_x - \dot{\sigma}_y = 0 \quad \text{and} \quad \dot{\tau} = 0, \quad \text{or} \quad \dot{\sigma}_x = \dot{\sigma}_y \quad \text{and} \quad \dot{\tau} = 0$$

When $\dot{\tau},_x$ and $\dot{\tau},_y$ vanish, the force equilibrium equations yield $\dot{\sigma}_x,_x$ and $\dot{\sigma}_y,_y$ equal to zero. Consequently, $\dot{\sigma}_x = \dot{\sigma}_y =$ constant and $\dot{\tau} = 0$ and the requirement that BC#1 requires $\dot{X}$ and/or $\dot{Y} = 0$ on part or all of the boundary leads to a contradiction unless,

$$\dot{\sigma}_x = \dot{\sigma}_y = \dot{\tau} = 0 \qquad \text{in the entire body at incipient plastic flow}$$

Proof of second auxiliary theorem:

$$I \equiv \left(\sigma_x - \sigma_x{}^o\right)\dot{\varepsilon}_x{}'' + \left(\sigma_y - \sigma_y{}^o\right)\dot{\varepsilon}_y{}'' + \left(\tau - \tau^o\right)\dot{\gamma}''$$

Obviously, I = 0 in the actual elastic region, otherwise $\left(\sigma_x - \sigma_y\right)^2 + 4 \cdot \tau^2 = 4 \cdot k^2$

$$\dot{\varepsilon}_x{}'' = -\dot{\varepsilon}_y{}'' \quad \rightarrow \quad \dot{\varepsilon}_x{}'' = \tfrac{1}{2} \cdot \left(\dot{\varepsilon}_x{}'' - \dot{\varepsilon}_y{}''\right), \quad \dot{\varepsilon}_y{}'' = \tfrac{1}{2} \cdot \left(\dot{\varepsilon}_y{}'' - \dot{\varepsilon}_x{}''\right)$$

$$I = \tfrac{1}{2} \cdot \left[\left(\sigma_x - \sigma_x{}^o\right) - \left(\sigma_y - \sigma_y{}^o\right)\right]\left(\dot{\varepsilon}_x{}'' - \dot{\varepsilon}_y{}''\right) + \left(\tau - \tau^o\right)\dot{\gamma}''$$

$$= \frac{\lambda}{4 \cdot G} \cdot \left\{\left[\left(\sigma_x - \sigma_x{}^o\right) - \left(\sigma_y - \sigma_y{}^o\right)\right]\left(\sigma_x - \sigma_y\right) + 4 \cdot \left(\tau - \tau^o\right)\tau\right\}$$

$$\frac{4 \cdot G}{\lambda} \cdot I = 4 \cdot k^2 - \left(\sigma_x{}^o - \sigma_y{}^o\right)\left(\sigma_x - \sigma_y\right) - 4 \cdot \tau^o \cdot \tau$$

Use the Schwarz' Inequality, see formula sheet (Appendix 4), with,

$$a_x = \sigma_x - \sigma_y, \quad a_y = 2 \cdot \tau; \quad b_x = \sigma_x{}^o - \sigma_y{}^o, \quad b_y = 2 \cdot \tau^o$$

to obtain,

$$\left(\sigma_x{}^o - \sigma_y{}^o\right)\left(\sigma_x - \sigma_y\right) + 4\cdot\tau^o\cdot\tau \le 2\cdot k\cdot\sqrt{\left(\sigma_x{}^o - \sigma_y{}^o\right)^2 + 4\cdot\tau^{o^2}} \le 4\cdot k^2$$

then

$$\frac{4\cdot G}{\lambda}\cdot I \ge 0 \quad \text{and since } \frac{G}{\lambda} > 0, \quad I \ge 0$$

Proof of third auxiliary theorem:

$$\sigma_x\cdot\dot{\varepsilon}_x{}^* + \sigma_y\cdot\dot{\varepsilon}_y{}^* + \tau\cdot\dot{\gamma}^* = \tfrac{1}{2}\cdot\left(\sigma_x - \sigma_y\right)\cdot\left(\dot{\varepsilon}_x{}^* - \dot{\varepsilon}_y{}^*\right) + \tau\cdot\dot{\gamma}^*$$

Use Schwarz' Inequality with,

$$a_x = \tfrac{1}{2}\cdot\left(\sigma_x - \sigma_y\right), \quad a_y = \tau \quad ; \quad b_x = \tfrac{1}{2}\cdot\left(\dot{\varepsilon}_x{}^* - \dot{\varepsilon}_y{}^*\right), \quad b_y = \dot{\gamma}^*$$

to obtain,

$$k^2\cdot\left[\left(\dot{\varepsilon}_x{}^* - \dot{\varepsilon}_y{}^*\right)^2 + \dot{\gamma}^{*2}\right] \ge \tfrac{1}{2}\cdot\left(\sigma_x - \sigma_y\right)\cdot\left(\dot{\varepsilon}_x{}^* - \dot{\varepsilon}_y{}^*\right) + \tau\cdot\dot{\gamma}^*$$

so that,

$$\sigma_x\cdot\dot{\varepsilon}_x{}^* + \sigma_y\cdot\dot{\varepsilon}_y{}^* + \tau\cdot\dot{\gamma}^* \le k\cdot\dot{\Gamma}^* \quad \text{where} \quad \dot{\Gamma}^* = \sqrt{\left(\dot{\varepsilon}_x{}^* - \dot{\varepsilon}_y{}^*\right)^2 + \dot{\gamma}^{*2}}$$

Proof of first limit theorem (for m_s):

From the second auxiliary theorem,

$$\int_R\left[\left(\sigma_x - \sigma_x{}^o\right)\cdot\dot{\varepsilon}_x{}'' + \left(\sigma_y - \sigma_y{}^o\right)\cdot\dot{\varepsilon}_y{}'' + \left(\tau - \tau^o\right)\cdot\dot{\gamma}''\right]\cdot dA \ge 0$$

Since, at incipient plastic flow, $\dot{\sigma}_x = \dot{\sigma}_y = \dot{\tau} = 0$ (from first auxiliary theorem), Hooke's law shows that $\dot{\varepsilon}_x{}' = \dot{\varepsilon}_y{}' = \dot{\gamma}' = 0$. Therefore, in the above integral the plastic strains can be replaced by the total strains. Then the virtual work theorem applies. Note the actual stresses are in equilibrium with $S\cdot X$ and $S\cdot Y$ on BC#1. Also $\sigma_x{}^o, \sigma_y{}^o, \tau^o$ are in equilibrium with $m_s\cdot X$ and $m_s\cdot Y$. Therefore,

$$\left(S - m_s\right)\cdot\oint_B\left(X\cdot v_x + Y\cdot v_y\right)ds + \sum\int_{L_{hk}}\left(T^{(h,k)} - T^{o\,(h,k)}\right)\cdot\left(v_T{}^{(h)} - v_T{}^{(k)}\right)dl_{hk}$$

$S\cdot\oint_B\left(X\cdot v_x + Y\cdot v_y\right)ds > 0$ since the actual stresses do work on the body.

In addition $\int_{L_{hk}} T^{(h,k)} \cdot \left(v_T^{(k)} - v_T^{(h)}\right) dl_{hk} > 0$ and $T^{(h,k)} = k$ so that,

$$\int_{L_{hk}} T^{o\,(h,k)} \cdot \left(v_T^{(k)} - v_t^{(h)}\right) dl_{hk} \le \int_{L_{hk}} T^{(h,k)} \cdot \left(v_T^{(k)} - v_t^{(h)}\right) dl_{hk}$$

since $T^{o\,(h,k)} \le k$. Therefore,

$$S - m_s \ge \frac{\sum \int_{L_{hk}} \left(T^{(h,k)} - T^{o\,(h,k)}\right)\left(v_T^{(k)} - v_t^{(h)}\right) dl_{hk}}{\oint_B \left(X \cdot v_x + Y \cdot v_y\right) ds} \ge 0$$

This theorem is implemented by finding a stress field that is statically admissible, then the loads for this stress field are less than the actual load.

Proof of second limit theorem (for m_k):

From the third auxiliary theorem,

$$\int_R \left(\sigma_x \cdot \dot{\varepsilon}_x^* + \sigma_y \cdot \dot{\varepsilon}_y^* + \tau \cdot \dot{\gamma}^*\right) \cdot dA \le k \cdot \int_R \dot{\Gamma}^* \cdot dA \quad \text{where} \quad \dot{\Gamma}^* = \sqrt{\left(\dot{\varepsilon}_x^* - \dot{\varepsilon}_y^*\right)^2 + \dot{\gamma}^{*2}}$$

Use the virtual work integral and the fact that the actual stresses are in equilibrium with $S \cdot X$ and $S \cdot Y$ on BC#1.

$$S \cdot \oint_B \left(X \cdot v_x^* + Y \cdot v_y^*\right) ds + \sum \int_{L_{hk}} T^{(h,k)} \cdot \left(v_T^{*(h)} - v_T^{*(k)}\right) dl_{hk} \le k \cdot \int_R \dot{\Gamma}^* \cdot dA$$

As the first integral is greater than zero,

$$S \le \frac{\sum \int_{L_{hk}} T^{(h,k)} \cdot \left(v_T^{*(k)} - v_T^{*(h)}\right) dl_{hk} + k \cdot \int_R \dot{\Gamma}^* \cdot dA}{\oint_B \left(X \cdot v_x^* + Y \cdot v_y^*\right) ds}$$

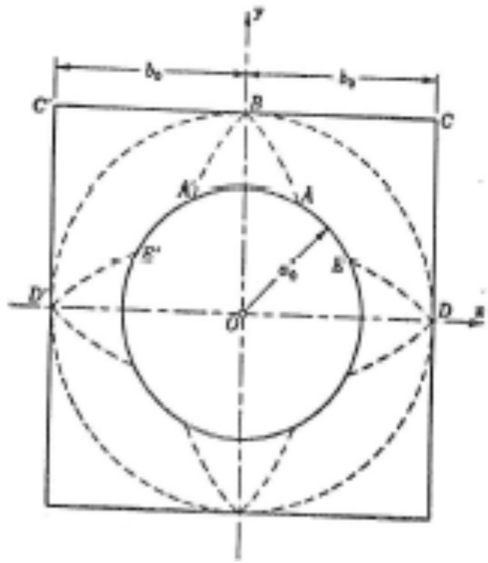
and the absolute value of $T^{(h,k)} \le k$ the bound on S may be written as.

$$S \le k \cdot \frac{\int_R \dot{\Gamma} \cdot dA + \sum \int_{L_{hk}} \left| v_t^{(k)} - v_t^{(h}\right| \cdot dl_{hk}}{\oint_B \left(X \cdot v_x + Y \cdot v_y\right) dS} \equiv m_k$$

P&H show that $m_k = S$ when v_x^* and v_y^* are the actual velocities.

This theorem is implemented by finding a kinematically admissible velocity field and making the integrations necessary to determine m_k. The loads $m_k \cdot X$ and $m_k \cdot Y$ are greater than the actual loads at incipient plastic flow, $S \cdot X$ and $S \cdot Y$.

As an illustration of the limit theorems a relatively interesting example follows.



The solution presented earlier for the elastic-plastic version of the Lame' solution gave the incipient plastic flow pressure, p^{**}, as,

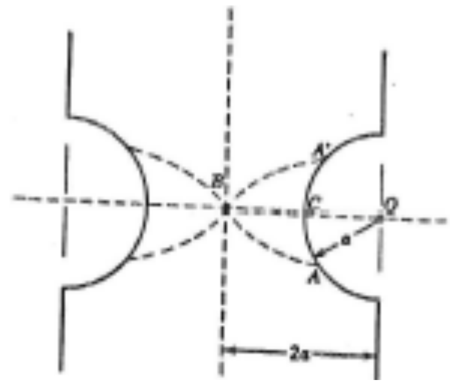
$$p^{**} = 2 \cdot k \cdot \ln\left(\frac{b}{a}\right)$$

Both a kinematically admissible and a statically admissible field are found using this solution. Also the stress field can be extended into the rigid part of the motion. And it turns out that,

$m_s = S = m_k$

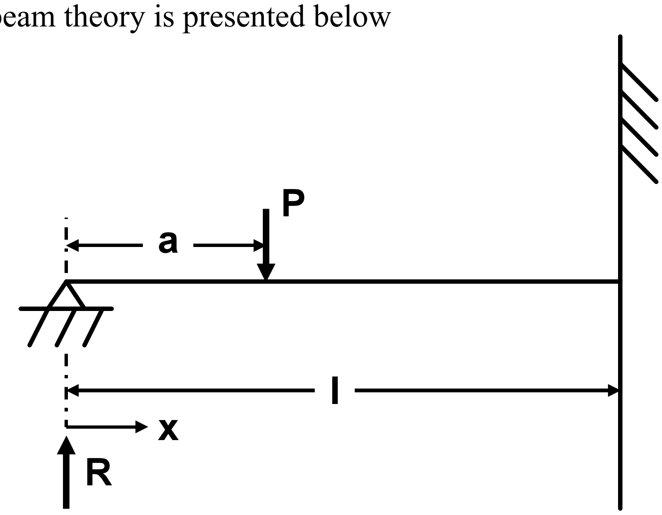
so the exact solution is p^{**}. This kind of scheme can be used to check solutions obtained by other methods.

Another illustrative solution that will be discussed is shown below.



The upper bound on T is improved using this kinematically admissible velocity field.

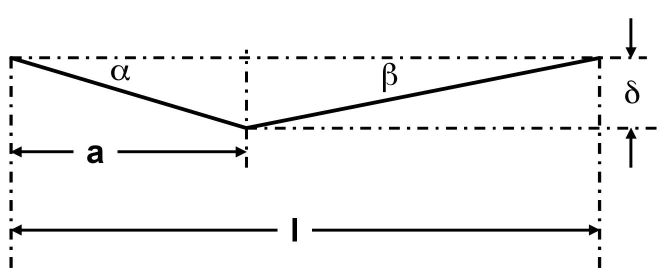
A simple example using beam theory is presented below



Statically admissible:

$$M(x) = R \cdot x - P_s \cdot \langle x - a \rangle$$
$$M(a) = M_O \ , \quad M(l) = -M_O$$
$$\rightarrow \quad P_s = \frac{M_O}{a} \cdot \frac{l + a}{l - a}$$

Kinematically admissible

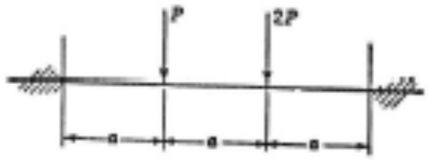


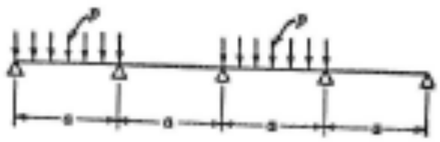
Take reference value of P as one, then m_k will equal the kinematically admissible incipient plastic flow load.

$$P_k = m_k = \frac{M_O \cdot (\alpha + \beta) + M_O \cdot \beta}{\delta}, \quad \alpha = \frac{\delta}{a}, \quad \beta = \frac{\delta}{l - a}, \quad \alpha + 2 \cdot \beta = \frac{\delta \cdot (l + a)}{a \cdot (l - a)}$$

$$\rightarrow \quad P_k = \frac{M_O}{a} \cdot \frac{l + a}{l - a}$$

The above problem was simple but contemplate two problems from P&H, Chapter 7, Problem 9 shown below.





P&H CHAPTER 8, EXTREMUM PRINCIPLES

No body forces
Elastic and plastic incompressibility
Assume continuity and continuous first derivatives of stress and velocity
Cartesian tensor formulation
A repeated Latin index in a term implies summation on that index
$,_x$ denotes partial differentiation with respect to x

The above leads to the following forms for some of the important equations

$\varepsilon_{ij} = \frac{1}{2}\cdot\left(u_{i,j} + u_{j,i}\right)$ note that $\gamma_{xy} = 2\cdot\varepsilon_{xy}$

$\sigma_{11} = \sigma_{xx}$ ... $\sigma_{12} = \tau_{xy}$

δ_{ij} = Kronecker delta = 1 if i = j , = 0 otherwise

$a_i \cdot \delta_{ij} = a_j$

$\sigma_{ij},_j = 0$ for force equilibrium equations

n_i = i^{th} component of unit length vector along exterior normal to surface element

T_i = i^{th} component of surface traction vector

$T_i = \sigma_{ij}\cdot n_j$

$s_{ij}\cdot s_{ij} = 2\cdot k^2$ von Mises yield condition

$s_{ij}\cdot \dot{s}_{ij} = 0$ no work hardening condition

$\dot{\varepsilon}_{ij} = \dfrac{1}{2\cdot G}\cdot\left(\dot{s}_{ij} + \lambda\cdot s_{ij}\right),\quad \lambda > 0,$ Prandtl – Reuss equations

$\dot{\varepsilon}_{ij} = \mu\cdot s_{ij},\quad \mu > 0,$ Mises equations

$\int_V F,_i \cdot dV = \oint_S F\cdot n_i \cdot dS$ F may be a scalar, vector or tensor

if $\sigma_{ij},_j = 0,\quad \varepsilon_{ij} = \frac{1}{2}\cdot\left(u_{i,j} + u_{j,i}\right)\quad \sigma_{ij} = \sigma_{ji}$

then

$$\int_V \sigma_{ij}\cdot\varepsilon_{ij}\cdot dV = \int_V \left(\sigma_{ij}\cdot u_i\right),_j \cdot dV = \oint_S \sigma_{ij}\cdot u_i \cdot n_j \cdot dS = \oint_S T_i \cdot u_i \cdot dS$$

Extremum Principles of the Prandtl-Reuss Theory (see Section 38 of P&H)

Definition for statically admissible stress rate field, $\dot{\sigma}_{ij}^{\,o}$,

1. $\dot{\sigma}_{ij}^{\,o},_j = 0$
2. $s_{ij}\cdot\dot{s}_{ij}^{\,o} \le 0$ whenever $s_{ij}\cdot s_{ij} = 2\cdot k^2$
3. BC#1 on S_T

Definition of a kinematically admissible strain rate field, $\dot{\varepsilon}_{ij}^{\,*}$,

1. Derived from a velocity field, v_i^*, that is incompressible

2. BC#2

Theorem 1. Among all statically admissible stress rate fields the actual one minimizes the expression,

$$K^{o} = \frac{1}{4 \cdot G} \cdot \int_{V} \dot{s}_{ij} \cdot \dot{s}_{ij} \cdot dV - \oint_{S_v} \dot{\sigma}_{ij}{}^{o} \cdot v_i \cdot n_j \cdot dS$$

Theorem 2. Among all kinematically admissible strain rate fields the actual one minimizes the expression,

$$L^{*} = \tfrac{1}{2} \cdot \int_{V} \dot{\sigma}_{ij}{}^{*} \cdot \dot{\varepsilon}_{ij}{}^{*} \cdot dV - \oint_{S_v} \dot{T}_i \cdot v_i{}^{*} \cdot dS, \quad \dot{\sigma}_{ij}{}^{*} \text{ found from } \dot{\varepsilon}_{ij}{}^{*} \text{ using Prandtl – Reuss equations}$$

Three Dimensional Limit Analysis (No discontinuities):

Statically admissible stress field, $\sigma_{ij}{}^{o}$,

1. $\sigma_{ij}{}^{o},_{j} = 0$
2. BC#1
3. $s_{ij} \cdot s_{ij} \leq 2 \cdot k^2$

Find m_s as largest value to satisfy the three conditions

Kinematically admissible field, v_i^{*},

1. Incompressible
2. BC#2, $v_i = 0$ on S_V
3. $\oint_{S} T_i \cdot v_i{}^{*} \cdot dS > 0$

Find m_k using,

$$m_k = \frac{\sqrt{2} \cdot k \cdot \int_{V} \sqrt{\dot{\varepsilon}_{ij}{}^{*} \cdot \dot{\varepsilon}_{ij}{}^{*}} \cdot dV}{\oint_{S} T_i \cdot v_i{}^{*} \cdot dS}, \quad \dot{\varepsilon}_{ij}{}^{*} = \tfrac{1}{2} \cdot \left(v_i{}^{*},_{j} + v_j{}^{*},_{i} \right)$$

Then $m_s \leq S \leq m_k$

Three Dimensional Application of Limit Analysis

The only application I am familiar with is for a Bingham fluid. Don Wood showed that the plasticity limit theorems apply to a Bingham fluid and he applied the theorems to the initiation of movement of a dense sphere (radius = R) suspended in a less dense Bingham fluid. The objective was to find bounds on the sphere density that initiates settling (cuttings in a drilling mud during cessation of drilling).

1965 PhD Thesis, Rice University,

Donald Bayne Wood, “Incipient Motion of a Spherical Body Suspended in a Bingham Material,” Mechanical Engineering Department, May 1965

The critical parameter, C, is

$$C = \frac{R \cdot \left(\gamma_{sphere} - \gamma_{fluid}\right)}{k}$$

Wood’s result $m_s = 5.75 \le C \le 12.25 = m_k$

A. N. Beris, J. A. Tsamopoulos, R. C. Armstrong and R. A. Brown, “Creeping Motion of a sphere Through a Bingham Plastic,” J. Fluid Mechanics (1985), vol. 158, pp 219-244

Their result, obtained numerically, is C = 10.49 which is pretty close to Wood’s kinematically admissible solution.

STARTING HILL'S TREATISE

Yield Surfaces

If the material is isotropic the yield surface must be expressible as a function of the invariants,

$f(J_1, J_2, J_3) = 0$

The invariants, in terms of the principle stresses. are,

$$J_1 = \sigma_1 + \sigma_2 + \sigma_3$$
$$J_2 = -\left(\sigma_1 \cdot \sigma_2 + \sigma_2 \cdot \sigma_3 + \sigma_3 \cdot \sigma_1\right)$$
$$J_3 = \sigma_1 \cdot \sigma_2 \cdot \sigma_3$$

Percy Bridgman, 1944, showed experimentally that a material's yield stress is, approximately, independent of hydrostatic pressure. Therefore, many formulations in plasticity theory are in terms of σ'_{ij} where,

$$\sigma'_{ij} = \sigma_{ij} - \sigma \cdot \delta_{ij}\ , \quad \sigma = \tfrac{1}{3} \cdot \sigma_{ii}$$

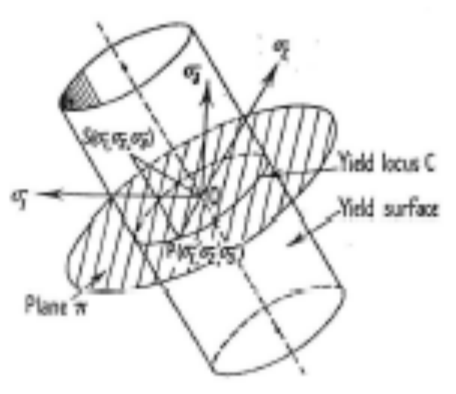
Then the yield surface is usually expressed as follows,

$f(J'_2, J'_3) = 0$

$$J'_2 = -\left(\sigma'_1 \cdot \sigma'_2 + \sigma'_2 \cdot \sigma'_3 + \sigma'_3 \cdot \sigma'_1\right) = \tfrac{1}{2} \cdot \left(\sigma'^2_1 + \sigma'^2_2 + \sigma'^2_3\right) = \tfrac{1}{2} \cdot \sigma'_{ij} \cdot \sigma'_{ij}$$
$$J'_3 = \sigma'_1 \cdot \sigma'_2 \cdot \sigma'_3 = \tfrac{1}{3} \cdot \left(\sigma'^3_1 + \sigma'^3_2 + \sigma'^3_3\right) = \tfrac{1}{3} \cdot \sigma'_{ij} \cdot \sigma'_{jk} \cdot \sigma'_{ki}$$

The Bauschinger effect is that if a material undergoes plastic flow for a particular load a subsequent reversed loading will induce yielding before the load magnitude is fully reversed to the yield value for the first yielding. Recalling the first structure that was analyzed in the series, it is not surprising that most materials exhibit a Bauschinger effect. This effect is neglected for the moment so that $f(J'_2, J'_3)$ must be an even function of J'_3.

It is instructive to visualize the yield surface in principle stress space. Consider the sketch below,

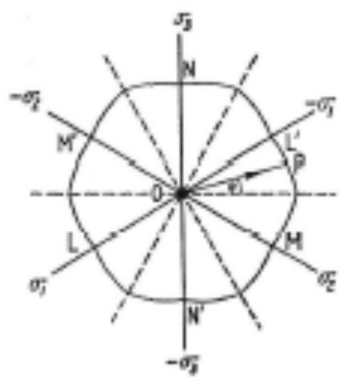


The dashed line makes equal angles with the Cartesian principle stress axes. Since the sum of the squares of the direction cosines is equal to one, these direction cosines are equal to $\frac{1}{\sqrt{3}}$ and the angle is 54.74^o. Plane Π is perpendicular to the dashed line and includes the origin. Therefore the angle between the Π plane and the principle stress axes is 90 – 54.74 = 35.26^o and the equation for the plane is,

$$\sigma_1 + \sigma_2 + \sigma_3 = 0$$

OS is the vector $(\sigma_1, \sigma_2, \sigma_3)$
OP is vector $(\sigma'_1, \sigma'_2, \sigma'_3)$ in the Π plane
PS is vector (σ, σ, σ) which is parallel to the dashed line and perpendicular to the Π plane
Clearly, the yield surface is a right cylinder and its intersection with the P plane defines the curve C

The sketch below is in the Π plane and C defines yield surface.



From isotropy of the yield surface C must be symmetric about LL', MM' and NN'
From no Bauschinger effect C must be symmetric about the perpendiculars to LL', MM' and NN' so that every 30^o section has the same shape.

Lode (1926) introduced a parameter, μ, that is helpful for designing some experiments.

$$\mu = \frac{2 \cdot \sigma_3 - \sigma_1 - \sigma_2}{\sigma_1 - \sigma_2} = -\sqrt{3} \cdot \tan(\theta)$$

If an investigator is certain the material being tested is isotropic with no Bauschinger effect then experiments are required only over the range $0 \le \theta \le 30^o$ $(0 \le \mu \le -1)$. To illustrate,

$$\mu = 0, \theta = 0, \sigma_3 = \tfrac{1}{2} \cdot (\sigma_1 + \sigma_2) \rightarrow (\text{pure shear}),\ \tfrac{1}{2} \cdot (\sigma_1 - \sigma_2, \sigma_2 - \sigma_1, 0) + \sigma = \tfrac{1}{2} \cdot (\sigma_1 + \sigma_2)$$
$$\mu = -1, \theta = 30^o, \sigma_2 = \sigma_3 \rightarrow (\text{uniaxial}),\ \tfrac{1}{2} \cdot (\sigma_1 - \sigma_2, 0, 0) + \sigma = \sigma_2$$

Possible experiments:

1. Combined torsion and tension of thin-walled tube

σ = tensile stress, τ = shear stress

$$\sigma_1 = \tfrac{1}{2}\cdot\sigma + \sqrt{\tfrac{1}{4}\cdot\sigma^2 + \tau^2}\ ,\quad \sigma_2 = \tfrac{1}{2}\cdot\sigma - \sqrt{\tfrac{1}{4}\cdot\sigma^2 + \tau^2}\ ,\quad \sigma_3 = 0$$

$$\mu = \frac{-\sigma}{\sqrt{\sigma^2 + 4\cdot\tau^2}}$$

2. Combined tension and pressure on thin-walled tube (neglect radial stress)

$$\sigma_1 = \frac{p\cdot r}{t}\ ,\quad \sigma_2 = 0\ ,\quad \sigma_3 = \frac{L}{2\cdot\pi\cdot r\cdot t}$$

$$\mu = \frac{L}{\pi\cdot r^2\cdot p} - 1$$

Two yield surfaces:

1. Tresca (1864)

   Order principle stresses so that $\sigma_1 \ge \sigma_2 \ge \sigma_3$ then yielding occurs when $\sigma_1 - \sigma_3$ =(twice the maximum shear stress) reaches a critical value. A more general form of the Tresca condition that is virtually never used is,

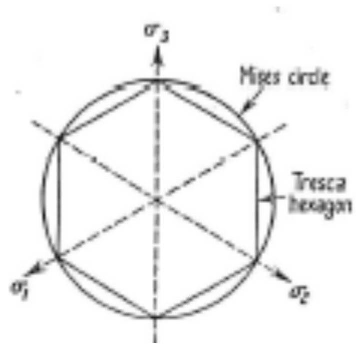
   $$4\cdot J'^{\,3}_2 - 27\cdot J'^{\,2}_3 - 36\cdot k^2\cdot J'^{\,2}_2 + 96\cdot k^4\cdot J'_2 - 64\cdot k^6 = 0$$

2. von Mises (1913)

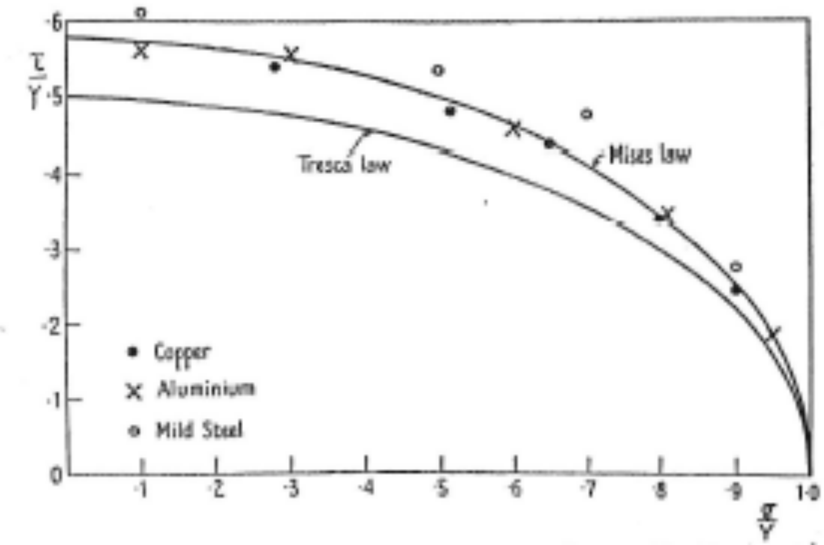
   $$J'_2 = \tfrac{1}{2}\cdot\sigma'_{ij}\cdot\sigma'_{ij} = k^2\ ,\quad \text{or}\quad (\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 = 6\cdot k$$

   For tensile test, yield point tensile stress = $\sqrt{3}\cdot k$ and k = shear stress yield point stress

When the two yield surfaces are compared, the yield points are usually adjusted so that the tensile yield point stress is the same from both criteria, see the figure below.



In 1931 Taylor and Quinney published a paper which is always cited as part of the comparison of the Tresca and von Mises yield surfaces. The sketch below shows the principal result.



Strain-Hardening, Standard Theory

1. Assume isotropy is preserved
2. Assume no matter by what strain path a given plastic stress state is reached, the final yield surface locus is the same. This is not consistent with dislocation theory. With this assumption a single infinity of distinct non-intersecting yield surface states exist. All of these states can be obtained by different amounts of pure tension. Therefore there is no Bauschinger effect.
3. All of the yield surfaces are circular in the Π plane so that the von Mises yield surface is chosen.

The radius of the yield surface in the Π plane = $\sqrt{3}\cdot k\cdot\cos\left(\frac{\pi}{2}-\arccos\left(\frac{1}{\sqrt{3}}\right)\right)=\sqrt{2}\cdot k$

Hooke's Law is $$d\varepsilon^{e}{}_{ij}=\frac{d\sigma'_{ij}}{2\cdot G}+(1-2\cdot\nu)\cdot\delta_{ij}\cdot\frac{d\sigma}{E}$$

$dW\equiv\sigma_{ij}\cdot d\varepsilon_{ij}=$ incremental external work per unit volume done on material during straining

$dW_P\equiv\sigma_{ij}\cdot(d\varepsilon_{ij}-d\varepsilon^{e}{}_{ij})\equiv\sigma_{ij}\cdot d\varepsilon^{P}{}_{ij}=$ incremental plastic work per unit volume

$W_P=\int\sigma_{ij}\cdot d\varepsilon^{P}{}_{ij}=$ plastic work per unit volume

A common hypothesis for the evaluation of the yield point in tension, $\bar{\sigma}$, that started as $\sqrt{3}\cdot k$ is,

$$\bar{\sigma}=\sqrt{\tfrac{1}{2}}\cdot\sqrt{(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2}=\sqrt{\tfrac{3}{2}}\cdot\sqrt{\sigma'_{ij}\cdot\sigma'_{ij}}=F(W_P)$$

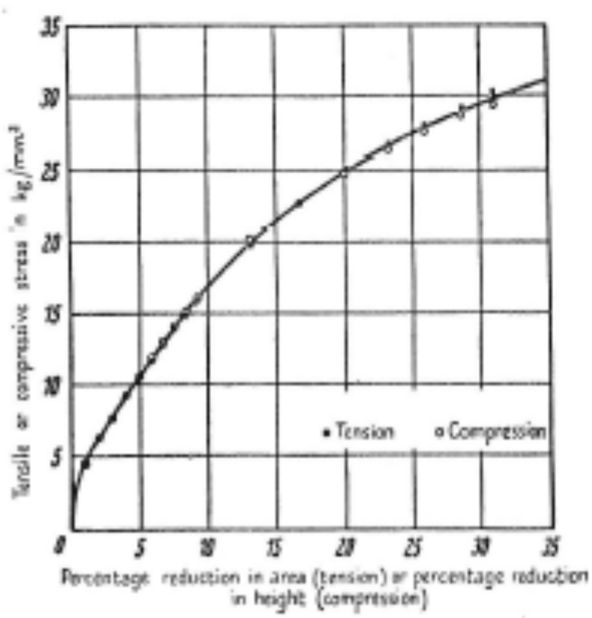
Comparison of stress-strain curves in tension, compression and torsion,

Tension: $\bar{\sigma} = F\left(\int_{l_0}^{l} \frac{\sigma \cdot dl}{l} - \frac{\sigma^2}{2 \cdot E}\right)$, plot σ versus $\ln\left(\frac{l}{l_0}\right)$, F is area under the curve up to σ

Compression: $p = F\left(\int_{h}^{h_0} p \cdot \frac{dh}{h} - \frac{p^2}{2 \cdot E}\right)$

Torsion: $\sqrt{3} \cdot \tau = F\left(\int_{0}^{\varphi} \tau \cdot d(\tan(\varphi)) - \frac{\tau^2}{2 \cdot G}\right)$

In 1925 Ludwik and Scheu published experimental results for tension and compression tests on annealed copper. The main result is shown in the figure below,



An alternative formulation appearing in the Literature is,

$\bar{\sigma} = H\left(\int d\bar{\varepsilon}^P\right)$, $d\bar{\varepsilon}^P = \sqrt{\frac{2}{3}} \cdot \sqrt{d\varepsilon^p_{ij} \cdot d\varepsilon^p_{ij}}$

Note that $d\bar{\sigma} = F' \cdot \bar{\sigma} \cdot d\bar{\varepsilon}^P = H' \cdot d\bar{\varepsilon}^P$ or $H' = F' \cdot \bar{\sigma}$

The Complete Stress-Strain Relations

Let $\bar{\mu}$ be the viscosity of a linear, incompressible fluid so that its constitutive formulation is,

$$\dot{\varepsilon}'_{ij} = \frac{\sigma'_{ij}}{2\bar{\mu}}, \quad \dot{\varepsilon}_{ii} = 0, \quad d\varepsilon'_{ij} = \frac{\sigma'_{ij}}{2\bar{\mu}} \cdot dt, \quad \text{so increment in } \varepsilon'_{ij} \text{ is } \propto \sigma'_{ij}$$

Metals have grains that are crystals. For a crystal, G. I. Taylor has shown that 5 shear slip planes are required for a general distortion. Dislocations move along slip planes and permit sliding at relatively low stress. Once started, no increase in stress is required to continue sliding (like a fluid). Unlike a fluid, grain boundaries stop the dislocations until they pile up and break across the boundary. This is related to work hardening. In view of this it is reasonable to assume the increment in ε'^{P}_{ij} is directly related to σ'_{ij}. The relations proposed by Lévy in 1871 and, independently, by von Mises 1913 express this concept in the form,

$$\frac{d\varepsilon^{P}_{ij}}{\sigma'_{ij}} = d\lambda, \quad d\lambda \text{ is not a constant, it is related to work hardening}$$

These equations direct $d\varepsilon^{P}_{ij}$ in the "direction" of σ'_{ij} Now, when the Lévy-Mises equations are satisfied,

$$dW_P = \sigma'_{ij} \cdot d\varepsilon^{P}_{ij} = \sigma'_{ij} \cdot \sigma'_{ij} \cdot d\lambda = \tfrac{2}{3} \cdot \bar{\sigma}^2 \cdot d\lambda = \bar{\sigma} \cdot d\bar{\varepsilon}^{P} = \frac{d\bar{\sigma}}{F'}, \quad \text{so that } d\lambda = \tfrac{3}{2} \cdot \frac{d\bar{\sigma}}{F' \cdot \bar{\sigma}^2} = \tfrac{3}{2} \cdot \frac{d\bar{\sigma}}{H' \cdot \bar{\sigma}}$$

and,

$$d\bar{\varepsilon}^{P} = \sqrt{\tfrac{2}{3}} \cdot \sqrt{d\varepsilon^{P}_{ij} \cdot d\varepsilon^{P}_{ij}} = \sqrt{\tfrac{2}{3}} \cdot \sqrt{\sigma'_{ij} \cdot \sigma'_{ij}} \cdot d\lambda = \tfrac{2}{3} \cdot \bar{\sigma} \cdot d\lambda$$

For a tension test,

$$[\sigma'] = \begin{bmatrix} \tfrac{2}{3}\times\sigma & 0 & 0 \\ 0 & -\tfrac{1}{3}\times\sigma & 0 \\ 0 & 0 & -\tfrac{1}{3}\times\sigma \end{bmatrix} \quad \text{and} \quad [d\varepsilon^{P}] = \begin{bmatrix} d\varepsilon^{P} & 0 & 0 \\ 0 & -\tfrac{1}{2}\times d\varepsilon^{P} & 0 \\ 0 & 0 & -\tfrac{1}{2}\times d\varepsilon^{P} \end{bmatrix}$$

$$\bar{\sigma} = \sqrt{\tfrac{3}{2}} \cdot \sqrt{\left(\tfrac{4}{9} + \tfrac{1}{9} + \tfrac{1}{9}\right)} \cdot \sigma = \sigma$$
$$d\bar{\varepsilon}^{P} = \sqrt{\tfrac{2}{3}} \cdot \sqrt{\left(1 + \tfrac{1}{4} + \tfrac{1}{4}\right)} \cdot d\varepsilon^{P} = d\varepsilon^{P}$$

Then a plot of σ versus ε^{P} from a tension test is the H function.

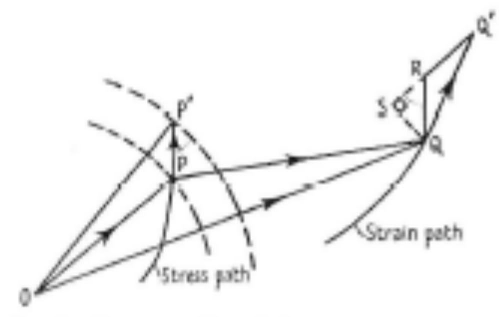
To summarize, when $d\bar{\sigma} > 0$,

$$d\varepsilon'_{ij} = \frac{d\sigma'_{ij}}{2 \cdot G} + \frac{3 \cdot \sigma'_{ij} \cdot d\bar{\sigma}}{2 \cdot \bar{\sigma} \cdot H'}$$
$$d\varepsilon_{ii} = \frac{1 - 2 \cdot \nu}{E}$$

A graphical representation of this stress-strain relation is possible on the Π plane but first write equations in the principle directions.

$$d\varepsilon'_1 = \frac{3\cdot\sigma'_1\cdot d\bar{\sigma}}{2\cdot\bar{\sigma}\cdot H'} + \frac{d\sigma'_1}{2\cdot G}$$
$$d\varepsilon'_2 = \frac{3\cdot\sigma'_2\cdot d\bar{\sigma}}{2\cdot\bar{\sigma}\cdot H'} + \frac{d\sigma'_2}{2\cdot G}$$
$$d\varepsilon'_3 = \frac{3\cdot\sigma'_3\cdot d\bar{\sigma}}{2\cdot\bar{\sigma}\cdot H'} + \frac{d\sigma'_3}{2\cdot G}$$

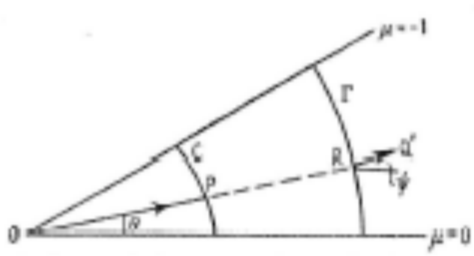
The figure below shows a Π plane with stress and strain superposed.



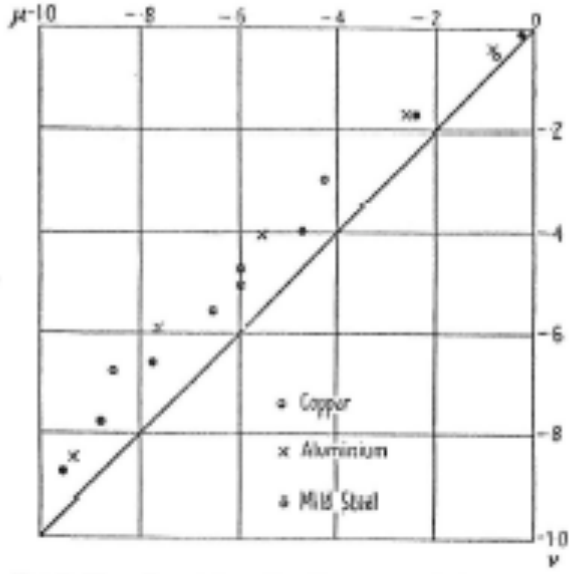
An experimental check on the above equations was reported in 1931 by Taylor and Quinney. The parameter ν was introduced with the definition,

$$\nu = -\sqrt{3}\cdot\tan(\psi) = \frac{2\cdot d\varepsilon^P{}_3 - d\varepsilon^P{}_1 - d\varepsilon^P{}_2}{d\varepsilon^P{}_1 - d\varepsilon^P{}_2}$$

The relationship between this parameter and the Lode variable, μ, introduced earlier is shown in the sketch below.



The results of the Taylor and Quinney experiments are shown in the figure below.



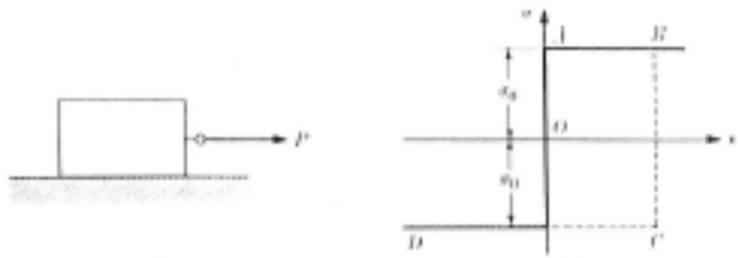
These results serve to remind investigators that the current theories are limited in their accuracy.

PLASTICITY WITH KINEMATICAL YIELD SURFACES

William Prager, *An Introduction to Plasticity*, Addison-Wesley Publishing Company, Inc., 1959, Chapter 1

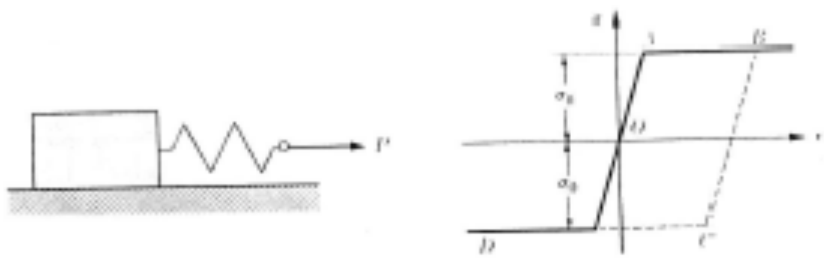
elastic

plastic



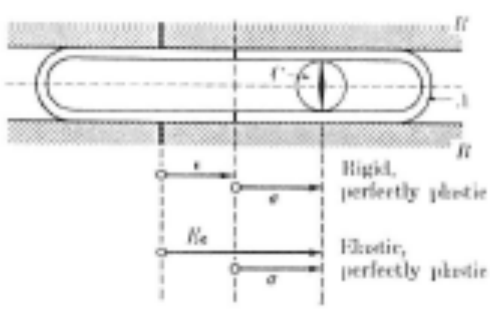
(a) (b)

elastic-plastic



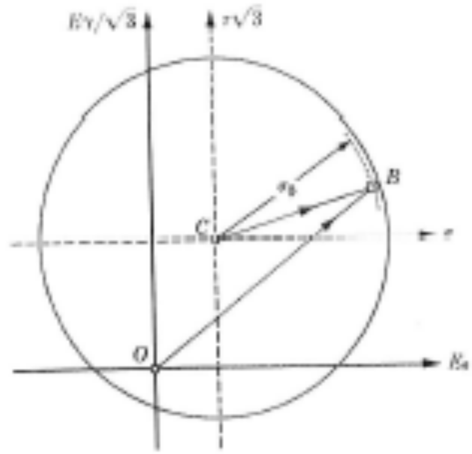
(a) (b)

kinematic hardening model



Combined tension and torsion of a thin-walled tube ($\nu = 0.5 \rightarrow 3 \cdot G = E$)

$$E \cdot \varepsilon^e = \sigma, \quad E \cdot \gamma^e = 3 \cdot \tau$$
$$\sigma^2 + 3 \cdot \tau^2 = {\sigma_0}^2$$
$$d\varepsilon^p = \lambda \cdot \sigma, \quad d\gamma^p = 3 \cdot \lambda \cdot \tau$$



$$\varepsilon = \varepsilon^e + \varepsilon^p, \quad \gamma = \gamma^e + \gamma^p$$
$$dW = \sigma \cdot d\varepsilon + \tau \cdot d\gamma$$

1. Vectors representing increments of stress and plastic strain are orthogonal so that the stress increment does no work on the increment of plastic strain.
2. Convexity

Introduce generalized forces, Q_i, and generalized displacements, q_i, so that the work increment, dW, is given by,

$$dW = Q_i \cdot dq_i, \quad dq_i = dq^e{}_i + dq^p{}_i$$

The yield surface is defined by the vanishing of a function $\Phi(Q_i)$ so that,

1. $\Phi(Q_i) = 0$ on the yield surface
2. $\Phi(Q_i) < 0$ inside the yield surface
3. $\Phi(Q_i) = 0$ must enclose point C

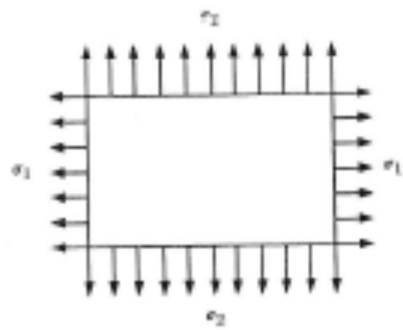
Consider $Q_i \rightarrow Q_i + dQ_i$ while $\Phi(Q_i) = 0$. This will be an elastic change and,

$$\frac{\partial \Phi}{\partial Q_i} \cdot dQ_i = 0$$

The vector, dQ_i, is tangent to the yield surface so the vector, $\frac{\partial \Phi}{\partial Q_i}$, is a vector in the outward normal direction to the yield surface. Therefore, the plastic flow law may be written as,

$$dq_i = \lambda \cdot \frac{\partial \Phi}{\partial Q_i} \qquad dW = Q_i \cdot dq_i = \lambda \cdot \left(Q_i \cdot \frac{\partial \Phi}{\partial Q_i} \right) \text{ must be } \geq 0, \therefore \lambda > 0$$

An illustrative example,



Von Mises: $\sigma_1^{\,2} + \sigma_2^{\,2} - \sigma_1 \cdot \sigma_2 - \sigma_0^{\,2} = 0$

Plastic flow law: $d\varepsilon^p{}_1 = \lambda \cdot (2 \cdot \sigma_1 - \sigma_2), \quad d\varepsilon^p{}_2 = \lambda \cdot (2 \cdot \sigma_2 - \sigma_1)$

Invert these: $\sigma_1 = \frac{1}{3\lambda} \cdot \left(2 \cdot d\varepsilon^p{}_1 + d\varepsilon^p{}_2\right), \quad \sigma_2 = \frac{1}{3\lambda} \cdot \left(2 \cdot d\varepsilon^p{}_2 + d\varepsilon^p{}_1\right)$

Substitute these stresses into the yield condition to obtain,

$$d\varepsilon^p{}_1{}^2 + d\varepsilon^p{}_2{}^2 + d\varepsilon^p{}_1 \cdot d\varepsilon^p{}_2 = 3 \cdot \sigma_0^{\,2} \cdot \lambda^2$$

Then,

$$\sigma_1 = \frac{\sigma_0}{\sqrt{3}} \cdot \frac{2 \cdot d\varepsilon^p{}_1 + d\varepsilon^p{}_2}{\sqrt{d\varepsilon^p{}_1{}^2 + d\varepsilon^p{}_2{}^2 + d\varepsilon^p{}_1 \cdot d\varepsilon^p{}_2}}$$

$$\sigma_2 = \frac{\sigma_0}{\sqrt{3}} \cdot \frac{2 \cdot d\varepsilon^p{}_2 + d\varepsilon^p{}_1}{\sqrt{d\varepsilon^p{}_1{}^2 + d\varepsilon^p{}_2{}^2 + d\varepsilon^p{}_1 \cdot d\varepsilon^p{}_2}}$$

$$dW^p = \frac{2 \cdot \sigma_0}{\sqrt{3}} \cdot \sqrt{d\varepsilon^p{}_1{}^2 + d\varepsilon^p{}_2{}^2 + d\varepsilon^p{}_1 \cdot d\varepsilon^p{}_2}$$

INTRODUCTION TO FRACTURE MECHANICS & FAILURE ASSESSMENT DIAGRAMS

1. BACKGROUND

The failure assessment diagram (FAD) results from a method used to combine plastic limit load design with design based on crack propagation models (fracture mechanics). This presentation is directed to explaining failure assessment diagrams as they are used in design. The initial part of this presentation considers only perfectly plastic materials with a yield point, σ_y. The plastic limit load is the load which will give unrestricted plastic flow.

If the material is sufficiently brittle, flaws in the material may cause fractures to propagate from the flaws before the limit load is reached. The usual procedure for designing for materials with flaws is to assume the maximum allowable flaw size has the most unfavorable shape and orientation. This is generally a crack whose length equals the flaw size and whose orientation is perpendicular to the maximum tensile stress. The study of the conditions under which the fracture will propagate is called “fracture mechanics”. The essential elements of fracture mechanics are given below.

The failure assessment diagram (FAD) may be constructed from the results of the plastic limit analysis and fracture mechanics. This construction is described below following the review of several fracture mechanics topics.

2. THE PHYSICAL BASIS OF FRACTURE MECHANICS

The primary assumption in fracture mechanics is that, under constant environmental conditions (e.g. temperature), a fixed amount of energy per unit area, Γ, is required to propagate a fracture. Γ is assumed to be a material property. The rationale for fracture propagation is that an advancing fracture causes energy to be released. When the energy release rate with respect to fracture area equals Γ a fracture will propagate. When the energy release rate with respect to fracture area is less than Γ a fracture will not propagate. Clearly, the energy release rate cannot exceed Γ.

A.A. Griffith[1] was the first investigator to propose the energy criterion for fracture. For the classical case of a crack of length $2 \cdot a$ imbedded in an infinite, elastic, thin sheet subjected to the far-field tension, σ_∞, perpendicular to the crack, Griffith used the plane stress solution by Inglis[2] to determine that,

$$\text{Rate of change of elastic energy with respect to a} = \frac{2 \cdot \pi \cdot \sigma_\infty^{\;2} \cdot a}{E} \qquad 2.1$$

where E is Young's modulus. This rate of change is set equal to the energy required to create a new surface per unit area, $2 \cdot \Gamma$, so that,

$$\Gamma = \frac{\pi \cdot \sigma_\infty^{\;2} \cdot a}{E} \qquad 2.2$$

This equation gives the condition, according to Griffith, that must be reached by an increasing σ_∞ in order for a fracture to propagate. Γ is regarded as a material property and it is called the "elastic energy release rate per crack tip". For the case of plane strain, E must be replaced by $\frac{E}{1 - \nu^2}$ where ν is Poisson's ratio.

G.R. Irwin[3,4,5] pointed out that part of the energy for propagating a crack was in the form of plastic deformation in the vicinity of the crack tip. Irwin made several contributions to improve upon and elucidate the work of Griffith.

The last equation was generalized by replacing the term $\pi \cdot \sigma_\infty^{\;2} \cdot a$ by the square of the stress intensity factor, K_I^2 for the crack. The plane strain condition may now be written as,

$$\Gamma = \left(1 - \nu^2\right) \frac{K_I^{\;2}}{E} \qquad 2.3$$

Further contributions to the development of contemporary fracture mechanics by D.S. Dugdale[6], G.I. Barenblatt[7], J.D. Eshelby[8] and J.R. Rice[9] are covered in the material presented below.

3. BRITTLE FRACTURE

When a fracture extends with very little plastic deformation (evaluated frequently by observing the fracture surface), the fracture process is described as "brittle" and elasticity theory is used to determine when a fracture will propagate.

For a two dimensional fracture loaded with a symmetric stress field with respect to the crack tip let,

$\kappa = 3 - 4 \cdot \upsilon$ for plane strain

3.1

$= \dfrac{3-\upsilon}{1+\upsilon}$ for plane stress

$$G = \text{shear modulus of elasticity} = \frac{E}{2 \cdot (1+\upsilon)} \qquad 3.2$$

where υ is Poisson's ratio and E is Young's modulus of elasticity. Then the stresses, $\sigma_{\alpha\beta}$, and the displacements, u_α, in the vicinity of the fracture tip in an elastic material may be written as,

$$\sigma_{\alpha\beta} = \frac{K_I}{\sqrt{2 \cdot \pi \cdot r}} \cdot \bar{\sigma}_{\alpha\beta}{}^{I}(\vartheta) \qquad 3.3$$

$$u_a = u_a{}^{0} + \frac{(1+\kappa) \cdot K_I}{2 \cdot G\pi} \cdot \sqrt{\frac{r}{2 \cdot}} \cdot \bar{u}_a{}^{I}(\vartheta, \kappa) \qquad 3.4$$

where, see sketch below,

K_I = a constant called the stress intensity factor

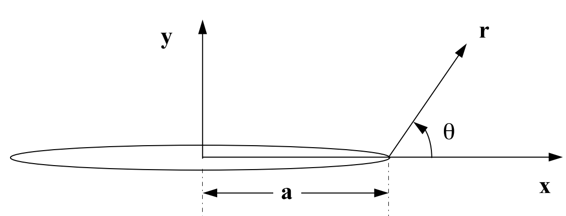
r = radial distance from the tip of the fracture

ϑ = angular position in the vicinity of the fracture tip

$\bar{\sigma}_{\alpha\beta}{}^{I}(\vartheta)$ = a bounded function of ϑ, (e.g. - $\bar{\sigma}_{22}{}^{I}(\vartheta=0)=1$)

$u_\alpha{}^{0}$ = an arbitrary displacement

$\bar{u}_\alpha{}^{I}(\vartheta, \kappa)$ = bounded function of ϑ and κ, (e.g. - $\bar{u}_2{}^{I}(\vartheta=0, \kappa) = 1$)



These results may be used with the condition that the far field displacements are held constant and there is incipient crack growth. Then,

$$\Gamma \cdot \Delta a = \frac{1}{2} \cdot \int_0^{\Delta a} \sigma_{22}(x,0) \cdot \left[u_2(x,0^+) - u_2(x,0^-) \right] dx \qquad 3.5$$

where Δa is arbitrarily small and with,

$$\sigma_{22}(x,0) = \frac{K_I(a)}{\sqrt{2 \cdot \pi \cdot x}} \quad 0 < x \le \Delta a \qquad 3.6$$

$$u_2(x,0^+) - u_2(x,0^-) = K_I(a+\Delta a) \cdot \frac{1+\kappa}{G} \cdot \sqrt{\frac{\Delta a - x}{2 \cdot \pi}} \qquad 3.7$$

the result is,

$$\Gamma \cdot \Delta a = \frac{1+\kappa}{4 \cdot \pi \cdot G} \cdot K_I(a) \cdot K_I(a+\Delta a) \cdot \int_0^{\Delta a} \sqrt{\frac{\Delta a - x}{x}} \cdot dx \qquad 3.8$$

The integral is evaluated as,

$$\int_0^{\Delta a} \sqrt{\frac{\Delta a - x}{x}} \cdot dx = \left[\sqrt{\Delta a - x} \cdot \sqrt{x} + \Delta a \cdot \sin^{-1}\left(\sqrt{\frac{x}{\Delta a}} \right) \right]_0^{\Delta a} = \frac{\pi}{2} \cdot \Delta a \qquad 3.9$$

so that as $\Delta a \rightarrow 0$,

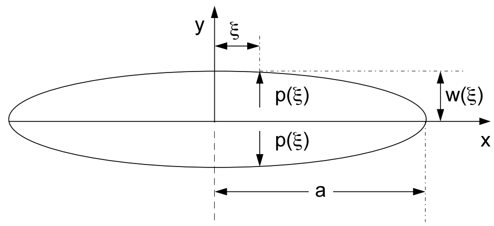
$$\Gamma = \frac{1+\kappa}{8} \cdot \frac{K_I^{\,2}}{G} \qquad 3.10$$

or

$$\Gamma = \begin{cases} \left(1 - \upsilon^2\right) \dfrac{K_I^{\,2}}{E} & \text{for plane strain} \\[2ex] \dfrac{K_I^{\,2}}{E} & \text{for plane stress} \end{cases} \qquad 3.11$$

The last two equations give the incipient brittle fracture propagation criteria for plane stress and plane strain conditions. These results are consistent with the results given in the preceding section.

As a consequence of the above results, there have been many elasticity problems solved to find values of K_I (and other stress intensity factors) for a variety of configurations and loads. One such solution is presented below for later use. The problem considered is shown in the sketch below,



It is the plane strain problem for a single crack of length 2a in an infinite two dimensional body subjected to a pressure distribution p(ξ) on the fracture face. The far-field stresses vanish. The only restriction on p(ξ) is that it be symmetric with respect to ξ, that is,

$$p(\xi) = p(-\xi) \qquad 3.12$$

The solution to this problem is in the literature. The stress intensity factor, K_I, at x = a, y = 0 is,

$$K_I = \frac{1}{\sqrt{\pi \cdot a}} \cdot \int_0^a p(\xi) \cdot \frac{2 \cdot a}{\sqrt{a^2 - \xi^2}} \cdot d\xi \qquad 3.13$$

and the fracture opening w(ξ) for each side of the fracture is,

$$w(x) = \frac{1-\upsilon}{G} \cdot \frac{1}{\pi} \cdot \int_{-a}^{a} p(\xi) \cdot \ln\left| \frac{\sqrt{a+\xi} \cdot \sqrt{a-x} + \sqrt{a-\xi} \cdot \sqrt{a+x}}{\sqrt{a+\xi} \cdot \sqrt{a-x} - \sqrt{a-\xi} \cdot \sqrt{a+x}} \right| \cdot d\xi \qquad 3.14$$

When p(ξ) is a constant, p_o, the above integrals give,

$$K_I = \frac{2 \cdot p_o \cdot a}{\sqrt{\pi \cdot a}} \cdot \int_0^a \frac{d\xi}{\sqrt{a^2 - \xi^2}} = p_o \cdot \sqrt{\pi \cdot a} \qquad 3.15$$

$$w(x) = (1-\upsilon) \cdot a \cdot \frac{p_o}{G} \cdot \sqrt{1 - \frac{x^2}{a^2}} = 2 \cdot \left(1 - \upsilon^2\right) a \cdot \frac{p_o}{E} \cdot \sqrt{1 - \frac{x^2}{a^2}} \qquad 3.16$$

When p(ξ) is zero in $0 \le \xi < c$ and the constant value p_1 in $c \le \xi \le a$ then,

$$K_I = p_1 \cdot \sqrt{\pi \cdot a} \cdot \left(1 - \frac{2}{\pi} \cdot \sin^{-1}\left(\frac{c}{a}\right)\right) \qquad 3.17$$

Comparisons of the elastic fracture mechanics propagation conditions with measured results for brittle materials show that the conditions are valid predictors of fracture propagation so long as there is little plastic flow. Early work in fracture mechanics was almost exclusively devoted to the kind of problems described above.

4. FRACTURE OF DUCTILE MATERIALS

The physical basis for fracture propagation in a ductile material is assumed to be the same as for a brittle material. The calculation of the energy release rate with respect to fracture area in the elastic-plastic case is more involved than the elastic case. It is therefore common, for practical calculations, to make an approximation which permits a great simplification in the calculations of the energy release rate. The approximation is that the material is non-linear elastic rather than plastic so that a strain energy density function, $W(\varepsilon_{ij})$, of the form,

$$\sigma_{ij} = \frac{\partial W}{\partial \varepsilon_{ij}} \qquad 4.1$$

exists. This approximation has the effect of introducing deformation plasticity theory (as opposed to incremental plasticity theory) if there is no local unloading. So long as the loading path is not too complex this approximation is acceptable. In this case the contour integral,

$$J = \oint_{\Gamma} \left(W \cdot n_x - \sigma_{ij} \cdot n_j \cdot \frac{\partial u_i}{\partial x} \right) \cdot ds \qquad 4.2$$

is independent of the contour enclosing the fracture tip. When x is the direction of fracture propagation, the J integral is also the energy release rate with respect to the fracture area (under the approximation of the existence of a strain energy density function). In this integral n_i is the component of $\vec{n}$ in the i^{th} direction and $\vec{n}$ is the outward unit length vector normal to the contour. The condition that the fracture propagates becomes,

$$J = \Gamma \qquad 4.3$$

For a specified configuration, stress-strain curve and loads the J integral may be readily evaluated numerically. Finite element programs such as MARC and ABAQUS have "built in" options for computing J integrals.

Note that in the linear elastic, plane strain case, the calculated values of the stress intensity factor and the J integral are related by,

$$\bar{J}_I = \left(1 - \nu^2\right) \frac{K_I^{\,2}}{E} \qquad 4.4$$

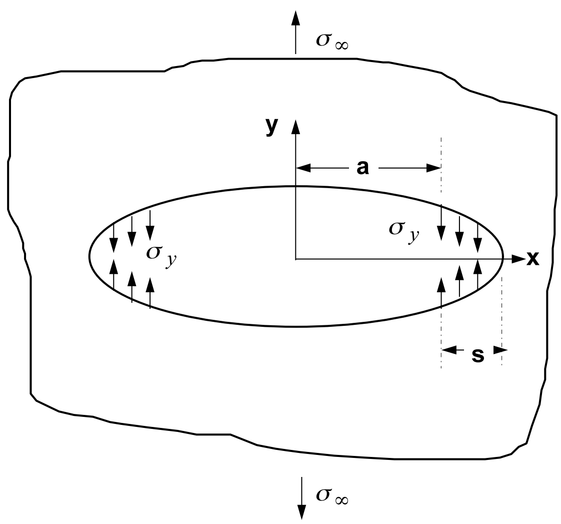
where the subscripted I refers to a Mode I crack loading and the bar indicates a J integral value for the *linear* elastic case. It is also convenient to define a material property, K_{Imat}, using,

$$\Gamma \equiv \left(1 - \nu^2\right) \frac{K_{Imat}^{\,2}}{E} \qquad 4.5$$

Therefore, K_{Imat} and Γ depend on the measured plastic behavior of the material while $\bar{J}_I$ and K_I are calculated using linear elastic material theory.

5. THE DUGDALE-BARENBLATT MODEL

This is a vastly simplified model of a fracture of length $2 \cdot a$ having plastic flow regions at its ends. The model is for a fracture in an infinite, thin sheet subjected to a far-field constant tension, σ_∞, perpendicular to the fracture. The material is assumed to be elastic-perfectly plastic with a yield point of σ_y. The *plane stress* model is sketched below,



The fracture is modeled as an *elastic* fracture of length 2(a + s) where the section of length s on each end has a tensile stress equal to σ_y acting on the surface. This stress is supposed to represent the stress acting in the material near the tip of the fracture. With this description and the solutions given earlier, the value of K_I at x = (a + s), y = 0 may be evaluated as,

$$K_I = \sigma_\infty \cdot \sqrt{\pi \cdot (a + s)} - \sigma_y \cdot \sqrt{\pi \cdot (a + s)} \cdot \left(1 - \frac{2}{\pi} \cdot \sin^{-1}\left(\frac{a}{a + s}\right)\right) \qquad 5.1$$

The condition,

$$K_I = 0 \qquad 5.2$$

is set so that no singularity will occur in the stress field at x = (a + s), y = 0. This condition has the effect of making the tips of the fracture cusp shaped (not shown this way in the figure above). The condition gives,

$$\frac{s}{a} = \sec\left(\frac{\pi}{2} \cdot \frac{\sigma_\infty}{\sigma_y}\right) - 1 \qquad 5.3$$

The tip of the crack being modeled is at x = a and the total crack tip opening displacement, CTOD, is determined at this point using the solution given above as,

$$CTOD = \frac{8}{\pi} \cdot \frac{\sigma_y}{E} \cdot a \cdot \ln\left[\sec\left(\frac{\pi}{2} \cdot \frac{\sigma_\infty}{\sigma_y}\right)\right] \qquad 5.4$$

Now the J integral is evaluated using a slender contour from the crack tip to x = (a + s) so that the contribution from $W \cdot n_x$ vanishes compared to the second term in the integrand of the J integral. Direct application of the above definition of the J integral noting that,

$$u_2(x = a + s, 0^+) = u_2(x = a + s, 0^-) = 0 \qquad 5.5$$

gives,

$$J = -\oint_\Gamma \sigma_y \cdot \left(\frac{\partial u_2}{\partial x} \cdot n_y \right) \cdot ds = -\sigma_y \cdot \int_a^{a+s} \left[\frac{\partial u_2(x, 0^+)}{\partial x} - \frac{\partial u_2(x, 0^-)}{\partial x} \right] \cdot dx$$

$$= \sigma_y \cdot CTOD = \frac{8}{\pi} \cdot \frac{{\sigma_y}^2}{E} \cdot a \cdot \ln\left[\sec\left(\frac{\pi}{2} \cdot \frac{\sigma_\infty}{\sigma_y} \right) \right] \qquad 5.6$$

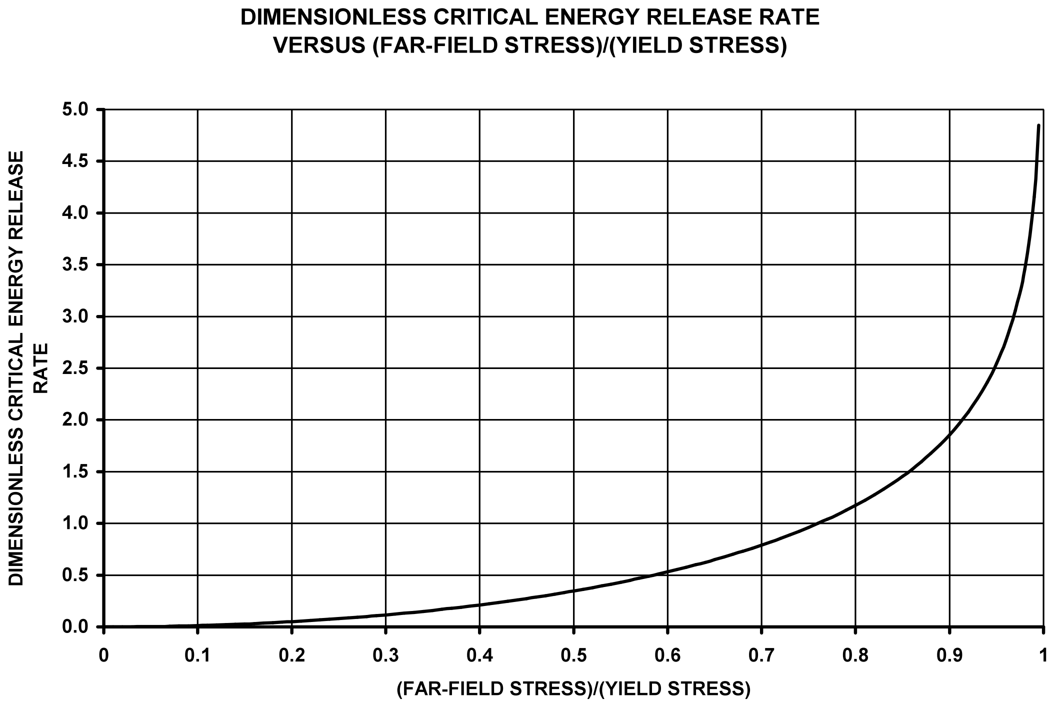
Note that this equation gives the value of the J integral at incipient crack propagation so for this case,

$$J = \Gamma \qquad 5.7$$

and, for incipient crack propagation,

$$\Gamma \cdot \frac{\pi \cdot E}{8 \cdot a \cdot {\sigma_y}^2} = \ln\left[\sec\left(\frac{\pi}{2} \cdot \frac{\sigma_\infty}{\sigma_y} \right) \right] \qquad 5.8$$

Therefore, when $\Gamma \cdot \frac{\pi \cdot E}{8 \cdot a \cdot {\sigma_y}^2}$, the dimensionless, critical energy release rate, is known [it is a function of material properties and crack (maximum flaw) size] then $\frac{\sigma_\infty}{\sigma_y}$ can be found for incipient crack propagation. For example, when $\frac{\sigma_\infty}{\sigma_y} = 0.5$, $\frac{\pi \cdot E \cdot \Gamma}{8 \cdot a \cdot {\sigma_y}^2} = 0.3465$. A plot representing this equation is given below.



The above figure shows that, as the applied stress increases relative to the yield stress, the dimensionless energy release rate required to prevent fracture growth increases rapidly. In practice, this last equation is usually put into another form. Define K_{Imat} and $\overline{K}_I$ as follows,

K_{Imat} = the material property Γ converted to stress intensity factor units $\equiv \sqrt{\Gamma \cdot E}$ 5.9

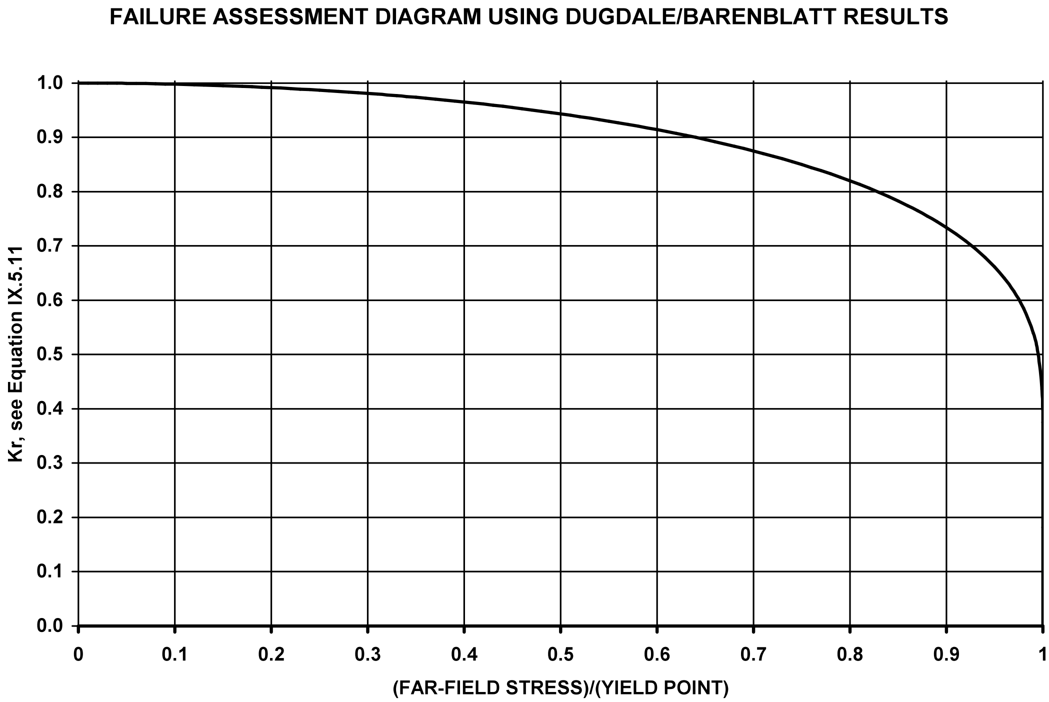
$\overline{K}_I =$ calculated elastic stress intensity factor for measured crack of length,

$$a = \sigma_\infty \cdot \sqrt{\pi \cdot a} \qquad 5.10$$

Now eliminate Γ and a from Equation 5.8 using Equations 5.9 and 5.10, respectively, to obtain,

$$Kr \equiv \frac{\overline{K}_I}{K_{Imat}} = \frac{(\text{elastic applied stress intensity})}{(\text{material allowable stress intensity})} = \frac{\sigma_\infty}{\sigma_y} \cdot \left[\frac{8}{\pi^2} \cdot \ln\left(\sec\left(\frac{\pi}{2} \cdot \frac{\sigma_\infty}{\sigma_y}\right)\right)\right]^{-\frac{1}{2}} \qquad 5.11$$

where Kr may be found when material properties Γ and E, the crack length a and the far-field stress σ_∞ are known. The crack propagation condition on the right-hand side of Equation 5.11 is a function of $\frac{\sigma_\infty}{\sigma_y}$ only. When this equation is plotted as shown below, the curve is in the form used in a failure assessment diagram.



Either of the above two curves can be used to determine a reduction of the allowable value of $\frac{\sigma_\infty}{\sigma_y}$ which is dependent on the flaw size. For example, an L80 steel is well approximated as a perfectly plastic material since the ratio of the ultimate strength to the yield point is about 1.1 and the elongation at failure is about 25%. Typical measured, material properties for L80 are,

Γ = 600. psi-in
σ_y = 100,000. psi
E = 30.E6 psi

The first of the above two curves shows that for $\frac{\sigma_\infty}{\sigma_y}$ to have a value of 0.97 (that is, little influence from fracture propagation) the value of the dimensionless critical energy release rate must be, at least, about 3. When $\Gamma \cdot \frac{\pi \cdot E}{8 \cdot a \cdot \sigma_y^{\ 2}}$ is set to 3.0 with the above material properties the value for the maximum flaw size, a, becomes,

a = 0.2356 inches

The Dugdale - Barenblatt model predicts that if inspection procedures can ensure that all flaws will have dimensions less than this value then the influence of fracture mechanics is small and the design can be based on plastic limit analysis.

6. THE LEVEL 2 FAILURE ASSESSMENT DIAGRAM FOR BURST

The second curve in the preceding section is based on a perfectly plastic material. This material model does not match many commonly used materials, such as K55, used in tubulars. The materials are typically ductile but the ratio of the ultimate tensile strength to the yield stress ranges up to about 1.6. While it is possible to use numerical methods (i.e. finite element methods) to create curves like those in Section 5 for work-hardening materials, it is not a practical way to proceed for day-to-day design. To overcome this difficulty and also include the fracture mechanics - plastic limit analysis transition in the design process, the failure assessment diagram has been introduced to design codes. These diagrams are "generalizations" of the last curve above. The value of the parameter $\mathrm{Kr} \equiv \frac{K_I}{K_{Imat}}$ is determined by a linear elastic solution and a material property measurement. The interpretation of $\frac{\sigma_\infty}{\sigma_y}$ is more subtle. The design codes are intended to include work-hardening materials. The derivation of the Dugdale-Barenblatt results presented in Section 5 is based on an elastic - perfectly plastic material in plane stress so that the stresses at the ends of the crack were set equal to σ_y. The strains near the tip of the crack are expected to be quite large so that it could be argued that the ultimate tensile strength, σ_{ULT}, should replace σ_y in Equation 5.11. In one code for Level 2 assessment, PD6493 (1991), this reasoning was used to replace σ_y by the, so called, flow stress, σ_f, where,

$$\sigma_f = \tfrac{1}{2} \cdot (\sigma_y + \sigma_{ULT}) \qquad 6.1$$

Let,

$$\mathrm{Sr} = \frac{\sigma_\infty}{\sigma_f} \qquad 6.2$$

so that,

$$\mathrm{Kr}\Big|_{PD6493\,(1991)} = \mathrm{Sr} \cdot \left[\frac{8}{\pi^2} \cdot \ln\left(\sec\left(\frac{\pi}{2} \cdot \mathrm{Sr} \right) \right) \right]^{-\frac{1}{2}} \qquad 6.3$$

A subsequent version PD6493 (1996) uses the above equation for Level 2 assessment of low hardening materials but uses an empirical equation for hardening materials.

An alternative approach to defining a FAD curve was published by the Electric Power Research Institute[10]. In this approach the Ramberg-Osgood stress-strain relationship was used to simulate real materials and find J integrals. The Ramberg-Osgood relations are,

$$\frac{\varepsilon}{\varepsilon_O} = \frac{\sigma}{\sigma_O} + \alpha \cdot \left(\frac{\sigma}{\sigma_O} \right)^n \qquad 6.4$$

where ε is strain, σ is stress; σ_O, α and n are constants and,

$$\varepsilon_O = \frac{\sigma_O}{E} \qquad 6.5$$

The two figures below are taken from Reference 11 and show typical results from the Electric Power Research Institute study. The strip yield model referred to in the figures is Equation 6.3.

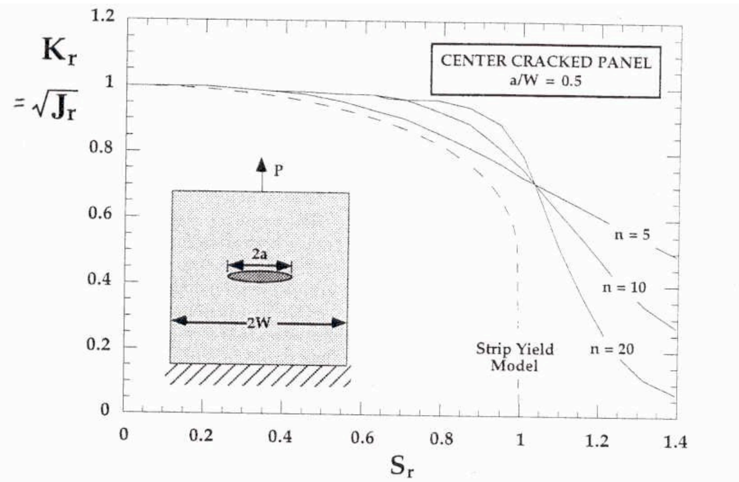


Fig. 22—Failure assessment diagrams for a center cracked panel with n = 5, 10, and 20 computed from the EPRI Handbook.

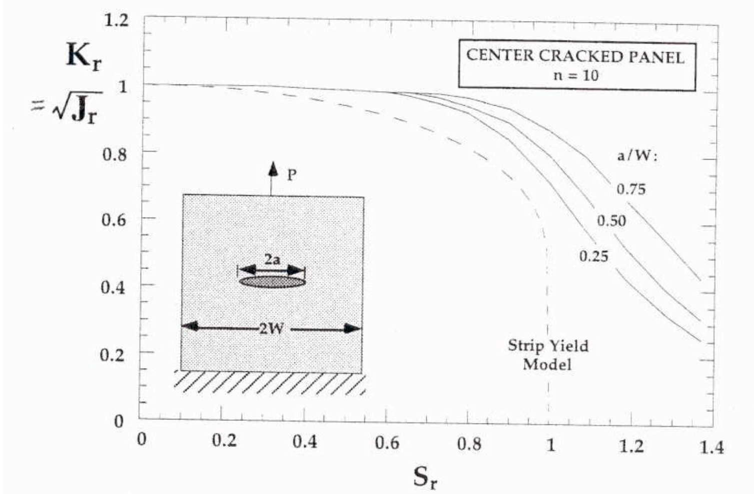


Fig. 23—EPRI J-based failure assessment diagrams for a center cracked panel with various a/W ratios.

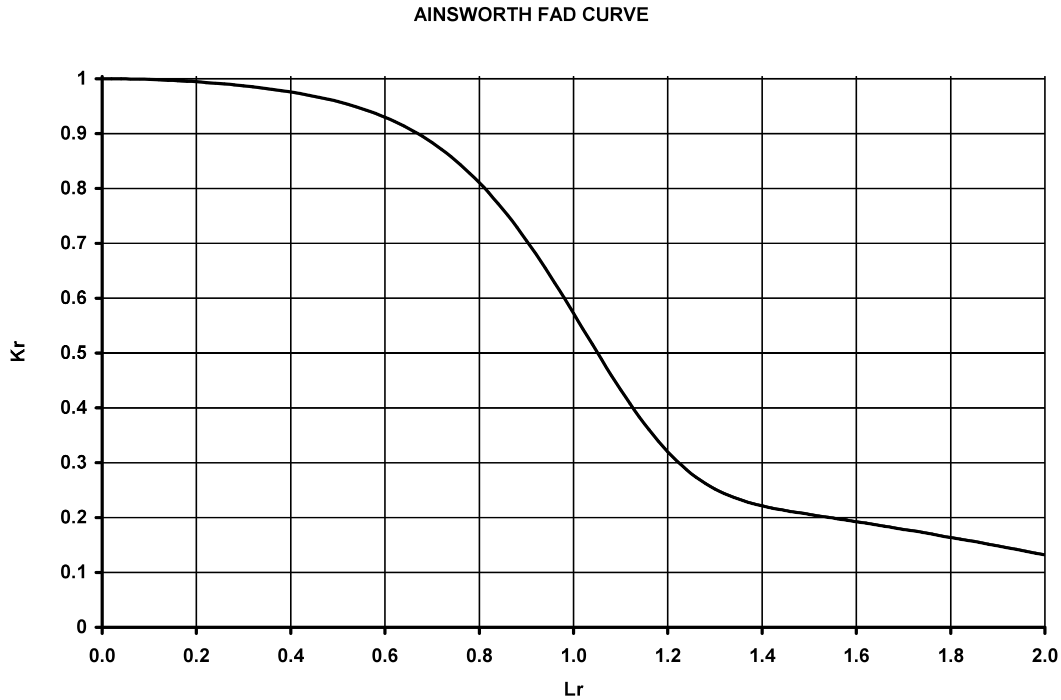
R.A. Ainsworth[12] developed some semi-empirical models that yield results similar to those in the above figures. One of these models has become rather popular and is included in PD6493 (1996) as well as API579. Let,

$$Lr = \frac{\sigma_\infty}{\sigma_y} \qquad 6.6$$

Then Ainsworth's model may be written as,

$$Kr = \left(1 - 0.14 \cdot \left(Lr^2\right)\right) \cdot \left(0.3 + 0.7 \cdot e^{-0.65 \cdot \left(Lr^6\right)}\right) \qquad 6.7$$

The failure assessment diagram associated with this equation is shown below.



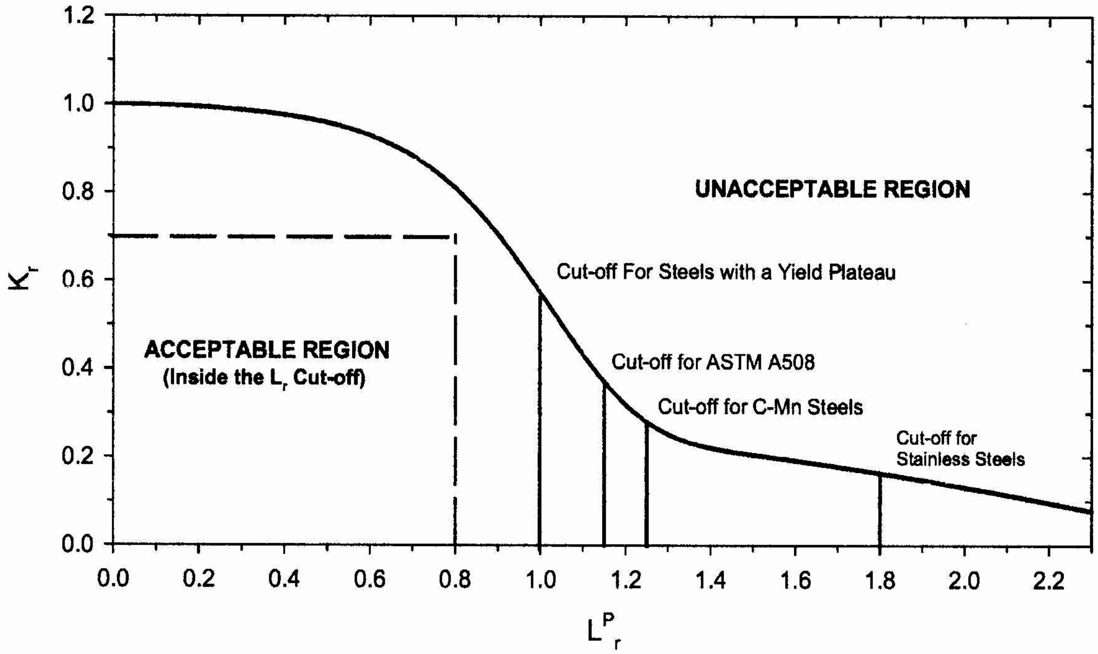
The design codes usually interpret Lr or Sr as,

$$\mathrm{Lr} = \frac{(\text{Applied Load})}{(\text{Limit Load based on } \sigma_y)} \quad \text{or} \quad \mathrm{Sr} = \frac{(\text{Applied Load})}{(\text{Limit Load based on } \sigma_f)} \qquad 6.7$$

Failure assessment diagrams are generally "adjusted" to account for the important parameters in particular design procedures. The above review gives an elementary introduction to the basis for these diagrams.

The next page is taken from the API579 design code. It shows how the FAD curves are treated in this code.

Figure 9.20
The Failure Assessment Diagram



Notes:

1. The FAD is defined using the following equation:

$$K_r = \left(1 - 0.14\left(L_r^P\right)^2\right)\left(0.3 + 0.7\exp\left[-0.65\left(L_r^P\right)^6\right]\right) \quad \text{for } L_r^P \le L_{r(\max)}^P \tag{9.46}$$

2. The extent of the FAD on the L_r^P axis is determined as follows:
   a. $L_{r(\max)}^P = 1.00$ for materials with yield point plateau (strain hardening exponent > 15),
   b. $L_{r(\max)}^P = 1.25$ for carbon-Mn steels,
   c. $L_{r(\max)}^P = 1.80$ for austenitic stainless steels, and
   d. $L_{r(\max)}^P = \dfrac{\sigma_f}{\sigma_{ys}}$ for other materials where σ_f is the flow stress (see Appendix F) and σ_{ys} is the yield stress; the flow stress and yield stress are evaluated at the assessment temperature.
3. If the strain hardening characteristics of the material are not known, then $L_{r(\max)}^P = 1.0$ should be used in the assessment.
4. The value of $L_{r(\max)}^P$ may be increased for redundant components (see Appendix D, paragraph D.2.5.2.b).
6. If $L_{r(\max)}^P = 1.0$, then the FAD may be defined using following equation:

$$K_r = \left(1.0 - \left(L_r^P\right)^{2.5}\right)^{0.20} \quad \text{for } L_{r(\max)}^P = 1.0 \tag{9.47}$$

6. The FAD in the dashed line is used with the assessment procedure in paragraph 9.4.3.3.

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COMMENTS CONCERNING FATIGUE FAILURE BASED ON FRACTURE MECHANICS

This is based on Chapter 10 of T. L. Anderson's *Fracture Mechanics*, Third Edition.

Prior to the introduction of fracture mechanics in fatigue life predictions, the S-N curve and the Goodman diagram were used for this purpose.

The two most successful uses of fracture mechanics have been for brittle material fracture and for fatigue life prediction.

The application of fracture mechanics results to fatigue life predictions in the case of repeated, identical, cyclic loadings is to predict the crack length, a, growth per cycle. The reciprocal of this prediction is integrated from an initial crack length of a_O to a final crack length of a_F to find the cycles, N, to fatigue failure. The experimental results required to establish the growth per cycle prediction has the form suggested below.

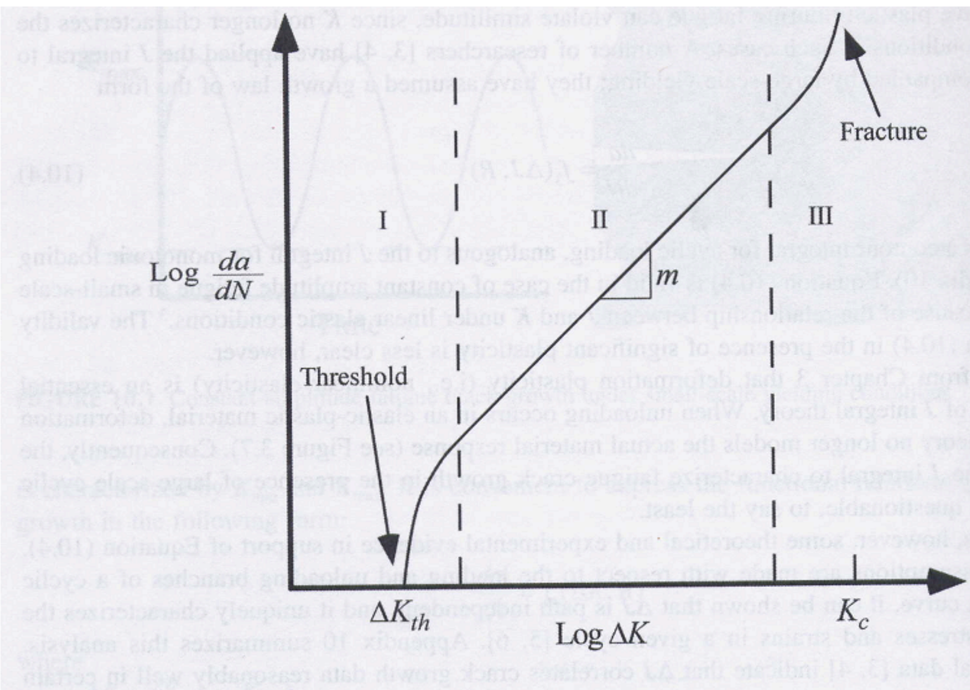


FIGURE 10.2 Typical fatigue crack growth behavior in metals.

where the load history (based on stress intensity factors) is sketched below.

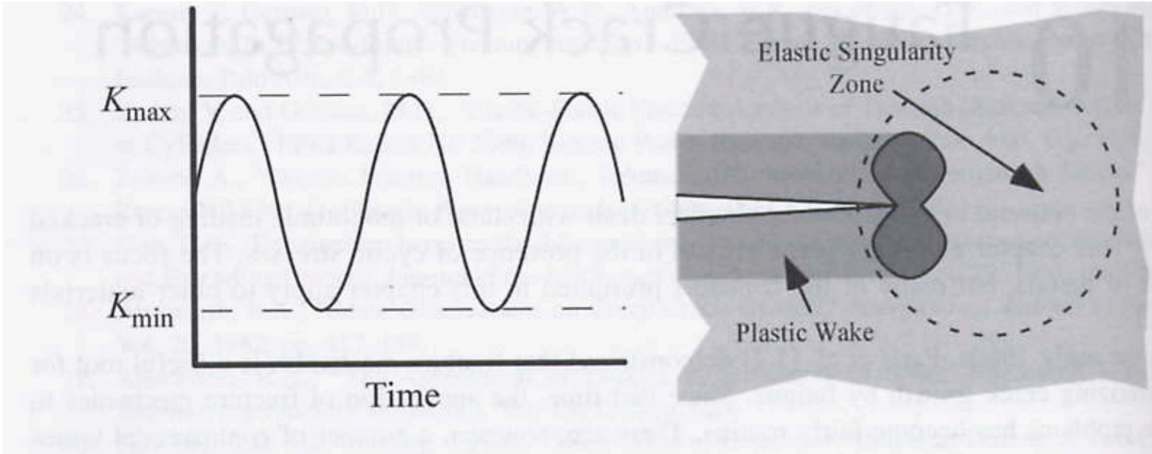


FIGURE 10.1 Constant amplitude fatigue crack growth under small-scale yielding conditions.

The physical behavior for these results is thought to be mathematically modeled using a growth rate equation of the following form,

$$\frac{da}{dN} = \text{crack length increase per cycle} = f_1(\Delta K, R), \quad \Delta K = K_{max} - K_{min}, \quad R = \frac{K_{min}}{K_{max}}$$

so that $K_{mean} = \frac{1}{2} \cdot K_{max} \cdot (1 + R)$

and $N = \text{number of cycles} = \int_{a_O}^{a_F} \frac{1}{\left(\frac{da}{dN}\right)} \cdot da$

When there as been previous cyclic loading, this past history must be taken into account.

For the central, straight-line section of the first figure, Paris and Erdogan (1960) proposed the following equation,

$$\frac{da}{dN} = C \cdot (\Delta K^m)$$

This equation is usually referred to as the *Paris Law*. Paris and Erdogan, based on available data, suggested that $m \cong 4$. Subsequent testing indicates that m lies between 2 and 4. Note that the last equation is independent of the ratio R and, therefore, the equation is not consistent with the Goodman diagram.

Other forms for fatigue life prediction may be found in Anderson's book. Several considerations apply to fatigue life predictions. Some of these are,

1. Crack closure can effectively decrease ΔK.
2. Crack wedging may be present because of corrosion
3. There are several proposed explanations for the existence of a threshold stress
4. The Miner-Palmgren criterion is used for this sort of calculation

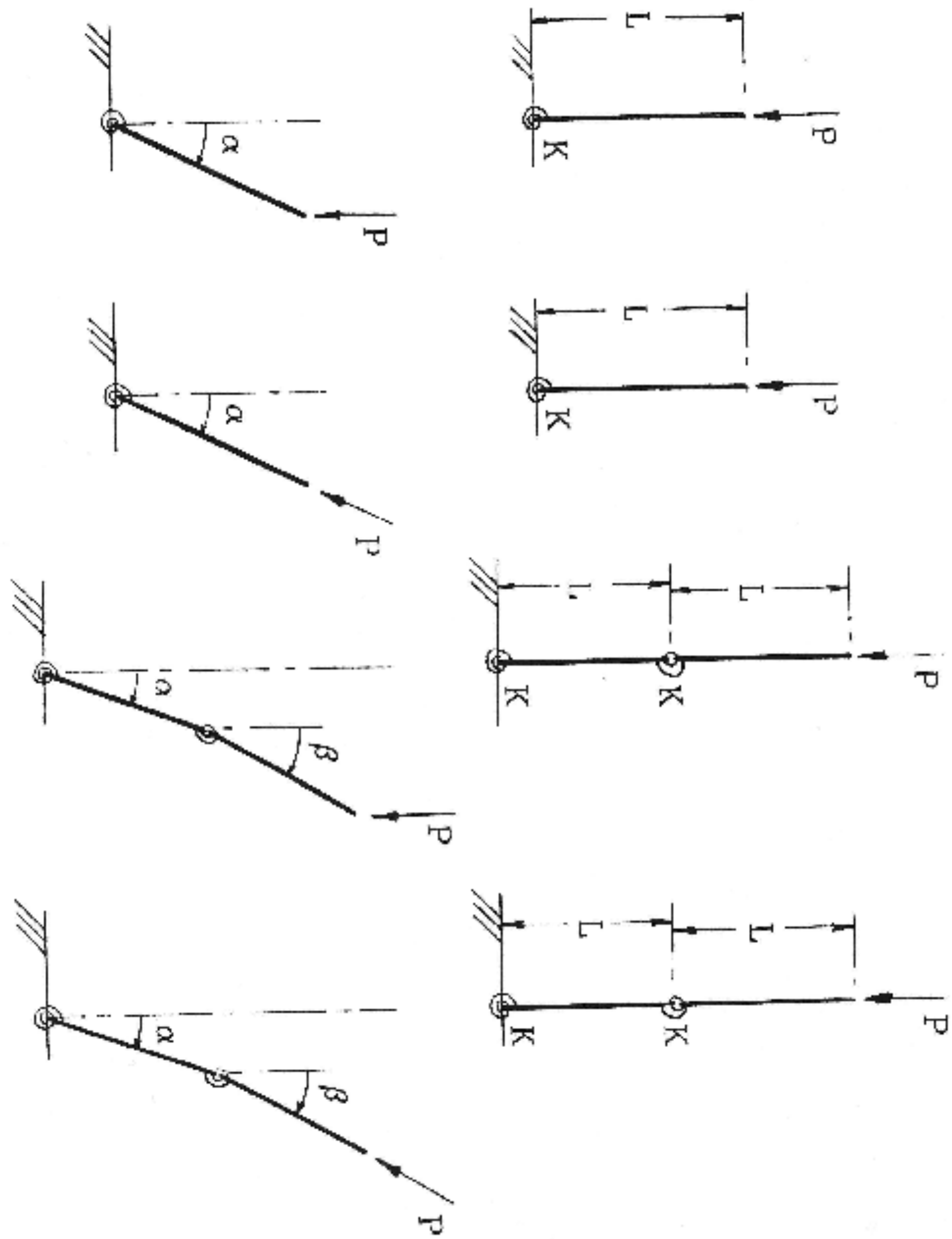
5. For low-cycle fatigue the classic L. F. Coffin and S. S. Manson formulation should be considered,

(plastic strain amplitude) = $\varepsilon_O \cdot (N^C)$

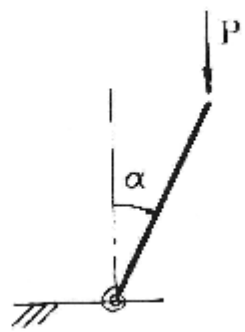
where ε_O is the strain to cause failure in one cycle, c is a constant for the material ($-0.5 > c > -0.7$) and N is the number of cycles to failure.

THE RELATION OF CLASSICAL MECHANICAL SYSTEM STABILITY THEORY TO DRUCKER'S POSTULATE

The purpose of this presentation is to show that, with a liberal interpretation, the usual mechanical system stability criteria are consistent with Drucker's Postulate. The film is rather comprehensive so only the slides are contained in this section. This material was presented first as a seminar in 1970 by Paul Paslay at Brown University to the Mechanics Group.


L
K
P
α
β

2



EQUILIBRIUM METHOD

$$P \cdot L \cdot \alpha - K \cdot \alpha = 0 \rightarrow P = K/L$$

ENERGY METHOD

$$PE = \frac{1}{2} \cdot K \cdot \alpha^2 - \frac{1}{2} \cdot P \cdot L \cdot \alpha^2$$

$$\delta PE = (K - P \cdot L) \cdot \alpha = 0 \rightarrow \alpha = 0$$

$$\delta^2 PE = K - P \cdot L > 0 \rightarrow P < K/L$$

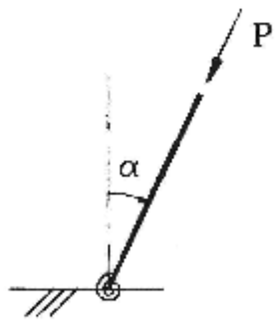
DYNAMIC METHOD

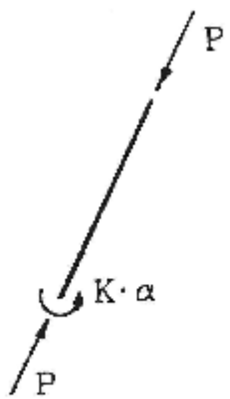
$$P \cdot L \cdot \alpha - K \cdot \alpha = IO \cdot \ddot{\alpha}$$

$$IO \cdot \ddot{\alpha} + (K - P \cdot L) \cdot \alpha = 0$$

$$\alpha = A \cdot e^{i \cdot \omega \cdot t} \rightarrow \omega^2 = (K - P \cdot L)/IO$$

stable if $P < K/L$





EQUILIBRIUM METHOD

$K \cdot \alpha = 0 \rightarrow \alpha = 0$, no buckling

ENERGY METHOD

$PE = \frac{1}{2} \cdot K \cdot \alpha^2$

$\delta PE = K \cdot \alpha \cdot \delta\alpha \rightarrow \alpha = 0$

$\delta^2 PE = \frac{1}{2} \cdot K \cdot (\delta\alpha)^2 \rightarrow$ stable

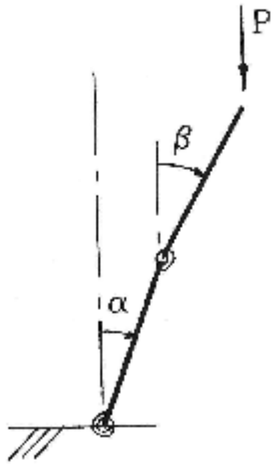
DYNAMIC METHOD

$-K \cdot \alpha = IO \cdot \ddot{\alpha}$

$IO \cdot \ddot{\alpha} + K \cdot \alpha = 0$

$\alpha = A \cdot e^{p \cdot t} \rightarrow p^2 = -K/IO$

stable



EQUILIBRIUM METHOD

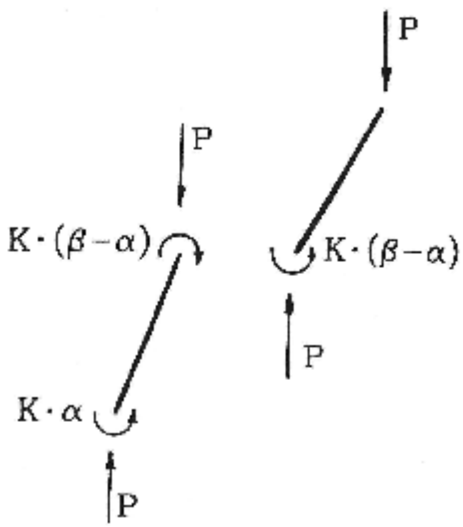
4

$$P \cdot L \cdot \alpha - K \cdot \alpha + K(\beta - \alpha) = 0$$

$$P \cdot L \cdot \beta - K(\beta - \alpha) = 0$$

$$\begin{vmatrix} P \cdot L - 2 \cdot K & K \\ K & P \cdot L - K \end{vmatrix} = 0 \rightarrow (P \cdot L/K)^2 - 3 \cdot (P \cdot L/K) + 1 = 0$$

$$P \cdot L/K = (3 - \sqrt{5})/2 \ , \quad \beta/\alpha = 2/(\sqrt{5} - 1) = 1.618$$



ENERGY METHOD

$$PE = \frac{1}{2} \cdot K \cdot (\alpha^2 + (\beta - \alpha)^2) - P \cdot L \cdot \left(\frac{1}{2} \cdot \alpha^2 + \frac{1}{2} \cdot \beta^2\right)$$

$$\delta PE = (2 \cdot K \cdot \alpha - K \cdot \beta - P \cdot L \cdot \alpha) \cdot \delta\alpha$$
$$+ (- K \cdot \alpha + K \cdot \beta - P \cdot L \cdot \beta) \cdot \delta\beta$$

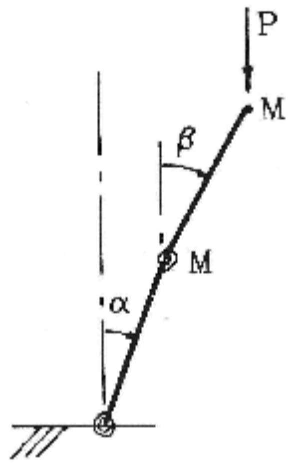
$$\delta^2 PE = \begin{bmatrix} \delta\alpha \\ \delta\beta \end{bmatrix}^T \cdot \begin{bmatrix} (K - P \cdot L/2) & K/2 \\ K/2 & (K - P \cdot L)/2 \end{bmatrix} \cdot \begin{bmatrix} \delta\alpha \\ \delta\beta \end{bmatrix}$$

$$K - P \cdot L/2 > 0 \qquad , \qquad P \cdot L/K < 2$$

$$(K - P \cdot L/2) \cdot (K - P \cdot L)/2 - K^2/4 > 0 \quad ,$$

$$(P \cdot L/K)^2 - 3 \cdot (P \cdot L/K) + 1 > 0$$
$$P \cdot L/K < (3 - \sqrt{5})/2$$

DYNAMIC METHOD



$$P1 = P2 = P$$

$$\left.\begin{aligned} PH &= M \cdot L \cdot (\ddot{\alpha} + \ddot{\beta}) \\ -PH + PH1 &= M \cdot L \cdot \ddot{\alpha} \end{aligned}\right\} PH1 = M \cdot L \cdot (2 \cdot \ddot{\alpha} + \ddot{\beta})$$

$$P \cdot L \cdot \alpha - K \cdot \alpha + K \cdot (\beta - \alpha) - PH \cdot L = M \cdot L^2 \cdot \ddot{\alpha}$$

$$P \cdot L \cdot \beta - K \cdot (\beta - \alpha) = M \cdot L^2 \cdot (\ddot{\alpha} + \ddot{\beta})$$

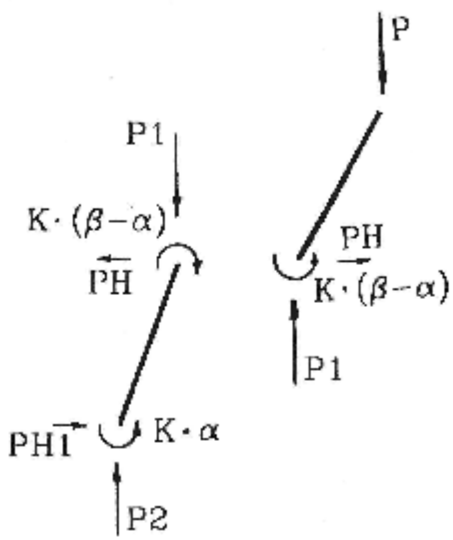
$$\alpha = A \cdot e^{i \cdot \omega \cdot t} \quad , \beta = B \cdot e^{i \cdot \omega \cdot t}$$

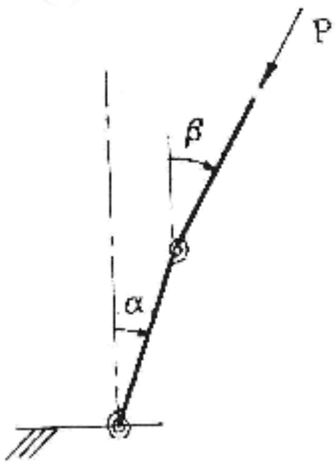
$$\begin{bmatrix} P \cdot L - 2 \cdot K + 2 \cdot \omega^2 \cdot M \cdot L^2 & K + \omega^2 \cdot M \cdot L^2 \\ K + \omega^2 \cdot M \cdot L^2 & P \cdot L - K + \omega^2 \cdot M \cdot L^2 \end{bmatrix} \cdot \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Let $\xi = P \cdot L/K \qquad \eta = \omega^2 \cdot M \cdot L^2/K$

$$2 \cdot \eta^2 + 3 \cdot (\xi - 2) \cdot \eta + (\xi^2 - 3 \cdot \xi + 1) = 0$$

η real & positive $\rightarrow \xi < \left(3 - \sqrt{5}\right)/2$





EQUILIBRIUM METHOD

$$\left.\begin{array}{l} K \cdot (\beta - \alpha) = 0 \rightarrow \alpha = \beta \\ K \cdot \alpha = 0 \rightarrow \alpha = 0 \end{array}\right\} \text{no alternative solution}$$

ENERGY METHOD

$$dW|_P = P \cdot L \cdot \left[\sin(\alpha - \beta) \cdot d\alpha + (0.) \cdot d\beta\right]$$

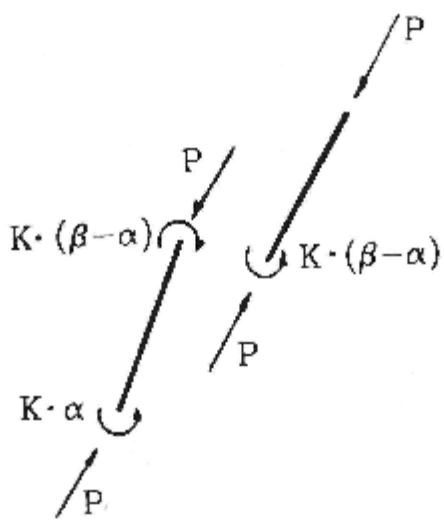
$$\frac{\partial}{\partial \beta}(\sin(\alpha - \beta)) \neq 0 \quad \therefore \quad P \text{ not representable by PE}$$

DYNAMIC METHOD [ADD M, PH, PH1]

$$\left.\begin{array}{l} PH = M \cdot L \cdot (\ddot{\alpha} + \ddot{\beta}) \\ PH1 - PH = M \cdot L \cdot \ddot{\alpha} \end{array}\right\} PH1 = M \cdot L \cdot (2 \cdot \ddot{\alpha} + \ddot{\beta})$$

$$K \cdot (\beta - \alpha) - K \cdot \alpha - PH \cdot L - P \cdot L \cdot (\beta - \alpha) = M \cdot L^2 \cdot \ddot{\alpha}$$

$$- K \cdot (\beta - \alpha) = M \cdot L^2 \cdot (\ddot{\alpha} + \ddot{\beta})$$



Eliminate PH from last two and set

$$\alpha = A \cdot e^{i \cdot w \cdot t}, \quad \beta = B \cdot e^{i \cdot w \cdot t}$$

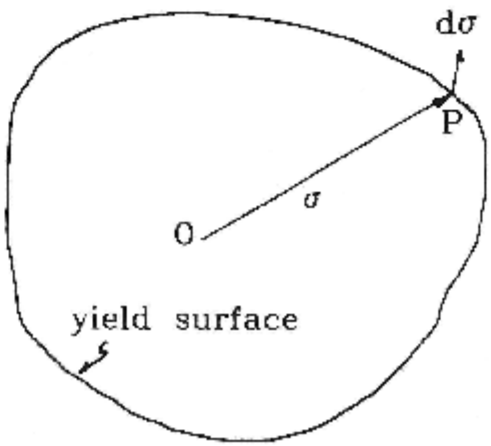
$$\begin{bmatrix} P \cdot L - 2 \cdot K + 2 \cdot \omega^2 \cdot M \cdot L^2 & -P \cdot L + K + \omega^2 \cdot M \cdot L^2 \\ K + \omega^2 \cdot M \cdot L^2 & -K + \omega^2 \cdot M \cdot L^2 \end{bmatrix} \cdot \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\xi = P \cdot L/K \;, \quad \eta = \omega^2 \cdot M \cdot L^2/K \;, \quad \eta^2 + (2 \cdot \xi - 6) \cdot \eta + 1 = 0$$

For real positive roots for η, $\quad \xi < 2$

$P < 2 \cdot K/L$

σ = stress



σ is on the yield surface, Point P

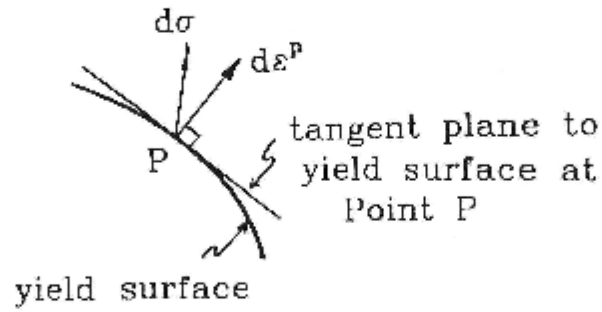
$d\sigma$ causes plastic straining

$d\varepsilon^P$ = increment of plastic strain resulting from $d\sigma$

DRUCKER'S STABILITY CRITERIA,

STABILITY IN THE LARGE: $\sigma \cdot d\varepsilon^P > 0$

STABILITY IN THE SMALL: $d\sigma \cdot d\varepsilon^P \geq 0$



STABILITY IN THE SMALL REQUIRES $d\varepsilon^P$ TO BE PERPENDICULAR TO THE TANGENT PLANE AT POINT P

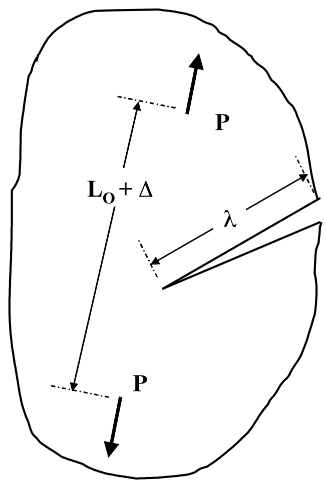
SUPERPOSED STRESS AND PLASTIC STRAIN SPACES

APPENDIX 1, ADDITIONAL FRACTURE MECHANICS CONSIDERATIONS

The following five sub-sections are attached to fill in some of the details in the body of the earlier report. The details are well established results that are found in fracture mechanics textbooks. The sole intention for including this appendix is to provide ready reference to points that may not be familiar to every reader.

1. THE RELATION OF ELASTIC ENERGY RELEASE RATE TO STRAIN ENERGY

Consider a two-dimensional linear elastic body with a single crack of length λ and a pair of self-equilibrating forces, P, as shown in the figure below. L_O is the distance between the load points when the loads are zero so that Δ is the elastic displacement on the load line. Assume the compliance, C(λ), between the loads is known so that the small displacement relation between P and λ is given by,



$$\Delta = C(\lambda) \cdot P \qquad \text{A1.1}$$

The strain energy associated with the loads on the body, U, is $\frac{1}{2} \cdot P \cdot \Delta$ and P may be eliminated to obtain,

$$U = \frac{1}{2 \cdot C(\lambda)} \cdot \Delta^2 \qquad \text{A1.2}$$

and,

$$\frac{\partial U}{\partial \lambda} = \frac{-\Delta^2}{2 \cdot C(\lambda)^2} \cdot \frac{\partial C(\lambda)}{\partial \lambda} \qquad \text{A1.3}$$

The Principle of Minimum Potential Energy will be employed to relate the strain energy to the energy release rate for an increasing crack length. The principle implies that the first variation of the potential energy, PE, vanish when the configuration (Δ) undergoes a variation under constant load conditions. The variation, δΔ, of the configuration is described, for present purposes, as $\Delta \rightarrow \Delta + \delta\Delta$ and $\lambda \rightarrow \lambda + \delta\lambda$ with δΔ and δλ being mathematically independent. The contribution to the potential energy change owing to δΔ is $-P \cdot \delta\Delta$. The contribution to the potential energy change owing to δλ is based on a fundamental postulate of fracture mechanics. The change in potential energy associated with δλ is set equal to **G** ˙ δ(crack area). The change in crack area is defined as t ˙ δλ where t is the thickness of the body. The quantity G, the energy release rate, is the energy expended to increase the surface area of the material per unit increased area. This energy release rate is postulated to be a material property so that it is the same for any problem so long as the material is not changed. This means that the potential energy may be written as,

$$\mathrm{PE} = \mathrm{U} - \mathrm{P}\cdot\Delta - \mathbf{G}\cdot \mathrm{t}\cdot\lambda = \frac{1}{2\cdot \mathrm{C}(\lambda)}\cdot\Delta^2 - \mathrm{P}\cdot\Delta - \mathbf{G}\cdot \mathrm{t}\cdot\lambda \qquad \text{A1.4}$$

and the first variation of the potential energy, which must vanish, becomes,

$$\delta\,\mathrm{PE} = \left(\frac{\Delta}{\mathrm{C}(\lambda)} - \mathrm{P}\right)\cdot\delta\Delta + \left(\frac{-\Delta^2}{2\cdot \mathrm{C}(\lambda)^2}\cdot\frac{\partial\,\mathrm{C}(\lambda)}{\partial\lambda} - \mathbf{G}\cdot \mathrm{t}\right)\cdot\delta\,\lambda = 0 \qquad \text{A1.5}$$

Since δΔ and δλ are independent, their multipliers must separately vanish so that,

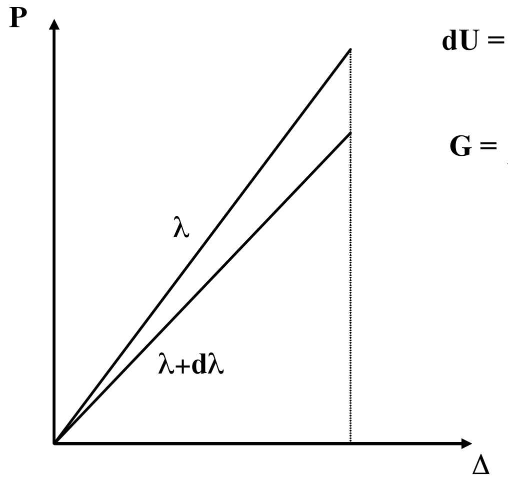
$$\mathrm{P} = \frac{\Delta}{\mathrm{C}(\lambda)} \qquad \text{(same as Equation A1.1)}$$

$$\mathbf{G} = \frac{-\Delta^2}{2\cdot \mathrm{t}\cdot \mathrm{C}(\lambda)^2}\cdot\frac{\partial\,\mathrm{C}(\lambda)}{\partial\lambda} \qquad \text{A1.6}$$

Combining Equations A1.3 and A1.6 to eliminate $\frac{\partial\,\mathrm{C}(\lambda)}{\partial\lambda}$ yields,

$$\mathbf{G} = \frac{1}{\mathrm{t}}\cdot\frac{\partial\,\mathrm{U}}{\partial\lambda} \qquad \text{A1.7}$$

Consequently, if a solution for the strain energy of a body containing a crack is known in terms of λ, then Equation A1.7 may be used to find **G** which is interpreted to be the energy release rate required to propagate the fracture (increase λ) at the imposed value of Δ. In general, **G** is determined from the value of Δ, determined experimentally, that initiates an increase of the measured crack length. The sketch below illustrates the influence of a small increase of crack length at a constant value of Δ. The lines labeled λ and λ+dλ are load-displacement curves for crack lengths of λ and λ+dλ, respectively.



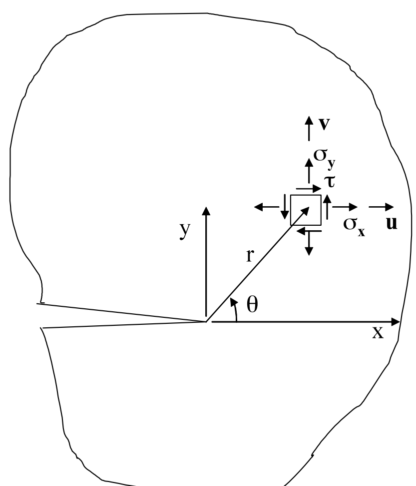
dU = area between λ and λ+dλ lines

$$G = \frac{dU}{t \cdot d\lambda}$$

It is important to note that the results in this section of the appendix apply only to linear elastic bodies. The results are therefore applicable to many brittle materials but not to elastic-plastic materials. The main use of Equation A1.7 is to determine the linear elastic stress intensity factor ($=\sqrt{E \cdot G}$, see section 2 of this appendix) for use in a failure assessment diagram, FAD.

2. THE RELATION OF ELASTIC STRESS INTENSITY FACTOR TO ELASTIC ENERGY RELEASE RATE

When a homogenous, isotropic, linear elastic body containing a crack is loaded, the resulting stresses and displacements may be found using the theory of elasticity. If the loading causes the crack to open the stresses and strains at the crack tip are singular (go to infinity). The sketch below of a crack in plane stress shows the coordinates x, y and r, θ, the displacements u and v and the stresses σ_x, σ_y and τ.



x,y origin at crack tip
x-direction parallel to crack

It is well established that the singularity causes stresses and displacements close to the crack tip (r sufficiently small) that may be written as,

$$\sigma_x = \sigma I_x + \sigma II_x$$
$$\sigma_y = \sigma I_y + \sigma II_y$$
$$\tau = \tau I + \tau II$$
$$u = uI + uII \qquad \text{A2.1-6}$$
$$v = vI + vII$$
$$G = \frac{E}{2 \cdot (1+\nu)}$$

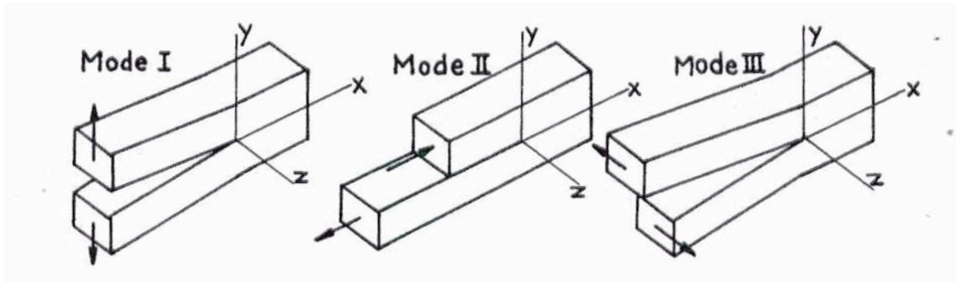
where,

$$\sigma I_x = \frac{K_I \cdot \cos\left(\frac{1}{2}\cdot\theta\right)}{\sqrt{2\cdot\pi\cdot r}} \cdot \left[1 - \sin\left(\frac{1}{2}\cdot\theta\right)\cdot \sin\left(\frac{3}{2}\cdot\theta\right)\right]$$
$$\sigma I_y = \frac{K_I \cdot \cos\left(\frac{1}{2}\cdot\theta\right)}{\sqrt{2\cdot\pi\cdot r}} \cdot \left[1 + \sin\left(\frac{1}{2}\cdot\theta\right)\cdot \sin\left(\frac{3}{2}\cdot\theta\right)\right]$$
$$\tau I = \frac{K_I \cdot \sin\left(\frac{1}{2}\cdot\theta\right)}{\sqrt{2\cdot\pi\cdot r}} \cdot \cos\left(\frac{1}{2}\cdot\theta\right)\cdot \cos\left(\frac{3}{2}\cdot\theta\right) \qquad \text{A2.7-11}$$
$$uI = \frac{K_I}{G}\cdot\sqrt{\frac{r}{2\cdot\pi}}\cdot\cos\left(\frac{1}{2}\cdot\theta\right)\cdot\left[1 - \frac{2\cdot\nu}{1+\nu} + \sin^2\left(\frac{1}{2}\cdot\theta\right)\right]$$
$$vI = \frac{K_I}{G}\cdot\sqrt{\frac{r}{2\cdot\pi}}\cdot\sin\left(\frac{1}{2}\cdot\theta\right)\cdot\left[2 - \frac{2\cdot\nu}{1+\nu} + \cos^2\left(\frac{1}{2}\cdot\theta\right)\right]$$

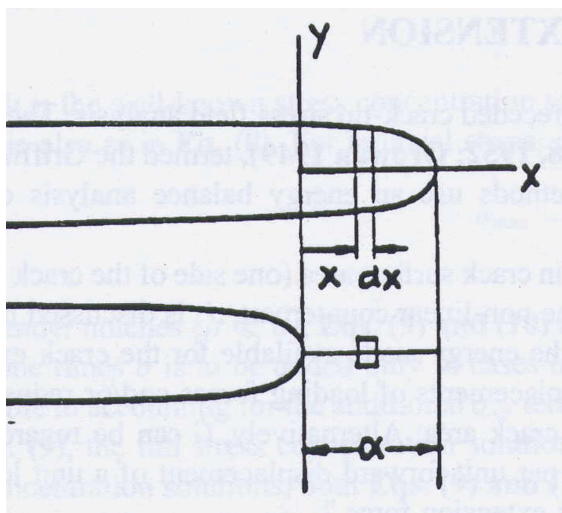
and,

$$\sigma II_x = -\frac{K_{II} \cdot \sin\left(\frac{1}{2}\cdot\theta\right)}{\sqrt{2\cdot\pi\cdot r}} \cdot \left[2 + \cos\left(\frac{1}{2}\cdot\theta\right)\cdot \cos\left(\frac{3}{2}\cdot\theta\right)\right]$$
$$\sigma II_y = \frac{K_{II} \cdot \sin\left(\frac{1}{2}\cdot\theta\right)}{\sqrt{2\cdot\pi\cdot r}} \cdot \cos\left(\frac{1}{2}\cdot\theta\right)\cdot \cos\left(\frac{3}{2}\cdot\theta\right)$$
$$\tau II = \frac{K_{II} \cdot \cos\left(\frac{1}{2}\cdot\theta\right)}{\sqrt{2\cdot\pi\cdot r}} \cdot \left[1 - \sin\left(\frac{1}{2}\cdot\theta\right)\cdot \sin\left(\frac{3}{2}\cdot\theta\right)\right] \qquad \text{A2.12-16}$$
$$uII = \frac{K_{II}}{G}\cdot\sqrt{\frac{r}{2\cdot\pi}}\cdot\sin\left(\frac{1}{2}\cdot\theta\right)\cdot\left[2 - \frac{2\cdot\nu}{1+\nu} + \cos^2\left(\frac{1}{2}\cdot\theta\right)\right]$$
$$vII = \frac{K_{II}}{G}\cdot\sqrt{\frac{r}{2\cdot\pi}}\cdot\cos\left(\frac{1}{2}\cdot\theta\right)\cdot\left[-1 + \frac{2\cdot\nu}{1+\nu} + \sin^2\left(\frac{1}{2}\cdot\theta\right)\right]$$

The stress intensity factors K_I and K_{II} correspond to different modes of deformation. When the three dimensional case (not considered here) is studied one more deformation mode is found. The three modes are shown in the sketch below. For the plane stress case being considered here the Mode III displacements cannot occur. All of the analyses in the body of this report are for Mode I singularities.



The case of Mode I displacements is now considered in order to relate K_I to G. That is, a plane stress case where $K_{II} = 0$ is considered. The figure below shows the case of a crack after propagation (upper) and before propagation (below). The amount of propagation is α, taken to be small enough for the stresses associated with the stress singularity to dominate. The change in strain energy per unit area, $\frac{\Delta U}{\Delta \text{area}}$, corresponding to the propagation may be calculated directly as,



$$\frac{\Delta U}{\Delta \text{area}} = \frac{2}{\alpha} \cdot \int_{x=0}^{x=\alpha} \left(\tfrac{1}{2} \cdot \sigma_y \Big|_{\substack{r=x \\ \theta=0}} \cdot v \Big|_{\substack{r=\alpha-x \\ \theta=\pi}} + \tfrac{1}{2} \cdot \tau \Big|_{\substack{r=x \\ \theta=0}} \cdot u \Big|_{\substack{r=\alpha-x \\ \theta=\pi}} \right) \cdot dx \qquad \text{A2.17}$$

When the stresses and displacements given in Equations A2.8-11 are substituted into Equation A2.17 the result is,

$$\frac{\Delta U}{\Delta \text{area}} = \frac{K_I^{\,2}}{\pi \cdot \alpha \cdot G \cdot (1+\nu)} \cdot \int_{x=0}^{x=\alpha} \sqrt{\frac{\alpha - x}{x}} \cdot dx = \frac{K_I^{\,2}}{\pi \cdot E} \cdot \left[\arcsin\left(\frac{2 \cdot x - \alpha}{\alpha} \right) \right]_{x=0}^{x=\alpha} = \frac{K_I^{\,2}}{E} \qquad \text{A2.18}$$

and, by definition, the left hand side of Equation A2.18 is the energy release rate, **G**, so that,

$$\mathbf{G} = \frac{K_I^{\,2}}{E} \qquad \text{A2.19}$$

Equation A2.19 is the desired relation between the elastic stress intensity factor, K_I, and the energy release rate, **G**. Using the same approach the following result can be established,

$$\mathbf{G} = \frac{K_I^{\,2}}{E} + \frac{K_{II}^{\,2}}{E} + \frac{(1+\nu)\cdot K_{III}^{\,2}}{E} \qquad \text{A2.20}$$

Note there are no cross products of the K's in Equation A2.20. In addition, when two loads, P_1 and P_2, contribute to K_I, K_{II} or K_{III} the stress intensity factors may be written, with obvious notation, as,

$$K_I(P_1, P_2) = K_I(P_1, 0) + K_I(0, P_2) \qquad \text{A2.21}$$

$$K_{II}(P_1, P_2) = K_{II}(P_1, 0) + K_{II}(0, P_2) \qquad \text{A2.22}$$

$$K_{III}(P_1, P_2) = K_{III}(P_1, 0) + K_{III}(0, P_2) \qquad \text{A2.23}$$

The last three equations are valid since the stresses are linear in the K's, see Equations A2.7-16.

3. THE J INTEGRAL AND ITS PATH INDEPENDENCE

The preceding results in this appendix have been limited to the case of linear elastic materials. The body of this report uses an approximate method to include plasticity influences. In the course of these analyses the J integral is determined. This integral measures the energy release rate for an approximation to elastic-plastic material behavior. The material limitations required to relate the J integral to the energy release rate are that the material be isotropic and homogeneous, the strains be small (compared to one) and the stress-strain behavior be derivable from an elastic strain energy function, $W(\varepsilon_{ij})$, so that,

W = $W(\varepsilon_{ij})$ = elastic strain energy function
ε_{ij} = i, j component of the strain tensor

$$= \tfrac{1}{2}\cdot\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \qquad \text{A3.1}$$

u_i = i^{th} displacement component, assumed to be twice differentiable with respect to x_i
x_i = i^{th} Cartesian coordinate
σ_{ij} = i, j component of the symmetric stress tensor

$$\equiv \frac{\partial W}{\partial \varepsilon_{ij}} \qquad \text{A3.2}$$

Although the material is, strictly speaking, elastic, the W function may be chosen so that the loading stress-strain curve resembles a typical measured stress-strain curve for an elastic-plastic material. So long as unloading does not occur and the ratios of stress components remain constant during loading, the formulation is a good representation for the elastic-plastic case.

The J integral value, J, is defined for current purposes (e.g. – no body forces) as follows for plane stress,

$$J = \int_{\Gamma} \left(W \cdot dy - \sum_{i} T_i \cdot \frac{\partial u_i}{\partial x} \cdot ds \right) \qquad \text{A3.3}$$

where the positive sense of integration is counterclockwise along the contour Γ in the body and,

Γ = contour in the material commencing on one face of the crack, enclosing the crack tip and ending on the other face of the crack
x = coordinate parallel to the straight crack direction
T_i = i^{th} component of surface traction (surface traction vector = force per unit surface area)
$= \sum_{j} \sigma_{ij} \cdot n_j$ A3.4
n_i = i^{th} component of unit outward normal to Γ contour
ds = differential arc length along Γ

With these definitions the coordinate changes along the Γ contour are,

$$dx = - n_y \cdot ds \qquad \text{A3.5}$$

$$dy = n_x \cdot ds \qquad \text{A3.6}$$

In the next section of this appendix the J integral will be shown to be equal to the energy release rate and this provides a physical connection of this integral to fracture mechanics. In this section an important property of the integral is derived. The property is that for a closed contour the integral is zero when there are no singularities inside or on the contour. This property can be used to show that the J integral is independent of Γ, that is, it is path independent for all contours satisfying the above definition of Γ.

The integrand in the J integral is now applied to a *closed* contour Γ_0 that contains no singularities inside or on its boundaries. This new integral is denoted by J_0. The derivation starts by eliminating T_i using Equations A3.3 and A3.4 with the result,

$$J_0 = \int_{\Gamma_0} \left(W \cdot dy - \sum_{i} \sum_{j} \sigma_{ij} \cdot \frac{\partial u_i}{\partial x} \cdot n_j \cdot ds \right) \qquad \text{A3.7}$$

In this equation dy may be replaced by $n_x \cdot ds$ and the order of summation changed to obtain,

$$J_0 = \int_{\Gamma_0} \left(W \cdot n_x \cdot ds - \sum_{j} \sum_{i} \left(\sigma_{ij} \cdot \frac{\partial u_i}{\partial x} \right) \cdot n_j \cdot ds \right) \qquad \text{A3.8}$$

The contour integrals in the above equation can be converted to area integrals by using the divergence theorem. This theorem is applicable only if the integrand contains no singularities (such as those appearing at the tip of the crack). Since Γ_0 satisfies this condition, the theorem is applicable to Equation A3.8. The divergence theorem is,

$$\int_{\Gamma_0} \varphi \cdot n_i \cdot ds = \int_{A_0} \frac{\partial \varphi}{\partial x_i} \cdot dA \qquad \text{A3.9}$$

where A_0 is the area enclosed by Γ_0. The result of applying this theorem to Equation A3.8 is,

$$J_0 = \int_{A_0} \left(\frac{\partial W}{\partial x} - \sum_j \sum_i \frac{\partial}{\partial x_j} \left(\sigma_{ij} \cdot \frac{\partial u_i}{\partial x} \right) \right) \cdot dx \cdot dy \qquad \text{A3.10}$$

Recall that $W = W(\varepsilon_{ij})$ so that the first term of the integrand may be expanded and Equation A3.2 may be used as follows,

$$\frac{\partial W}{\partial x} = \sum_i \sum_j \frac{\partial W}{\partial \varepsilon_{ij}} \cdot \frac{\partial \varepsilon_{ij}}{\partial x} = \sum_i \sum_j \tfrac{1}{2} \cdot \sigma_{ij} \cdot \left[\frac{\partial}{\partial x} \left(\frac{\partial u_i}{\partial x_j} \right) + \frac{\partial}{\partial x} \left(\frac{\partial u_j}{\partial x_i} \right) \right] \qquad \text{A3.11}$$

Take the second term in the square brackets in the above equation, interchange the i and j indices and recall that $\sigma_{ij} = \sigma_{ji}$ to obtain,

$$\frac{\partial W}{\partial x} = \sum_i \sum_j \sigma_{ij} \cdot \frac{\partial}{\partial x} \left(\frac{\partial u_i}{\partial x_j} \right) = \sum_i \sum_j \sigma_{ij} \cdot \frac{\partial}{\partial x_j} \left(\frac{\partial u_i}{\partial x} \right) \qquad \text{A3.12}$$

The force equilibrium equations (without body forces) are,

$$\sum_j \frac{\partial \sigma_{ij}}{\partial x_j} = 0 \qquad \text{A3.13}$$

and they may be used with Equation A3.12 to determine that,

$$\frac{\partial W}{\partial x} = \sum_i \sum_j \frac{\partial}{\partial x_j} \left(\sigma_{ij} \cdot \frac{\partial u_i}{\partial x} \right) \qquad \text{A3.14}$$

When Equations A3.10 and A3.14 are combined to eliminate $\frac{\partial W}{\partial x}$ the result is,

$$J_0 = 0 \qquad \text{A3.15}$$

The vanishing of J_0 can be used to establish the independence of the J integral. Consider the choice for the Γ_0 contour and sense of integration shown below. The closed contour is separated into four parts Γ_1, Γ_2, Γ_3 and Γ_4. Parts Γ_1 and Γ_3 are each legitimate contours for the J integral. Let the integral in Equation A3.7 be separated into parts J_1, J_2, J_3 and J_4 corresponding to Γ_1, Γ_2, Γ_3 and Γ_4. Since J_0 vanishes,

$$J_1 + J_2 + J_3 + J_4 = 0 \qquad \text{A3.16}$$

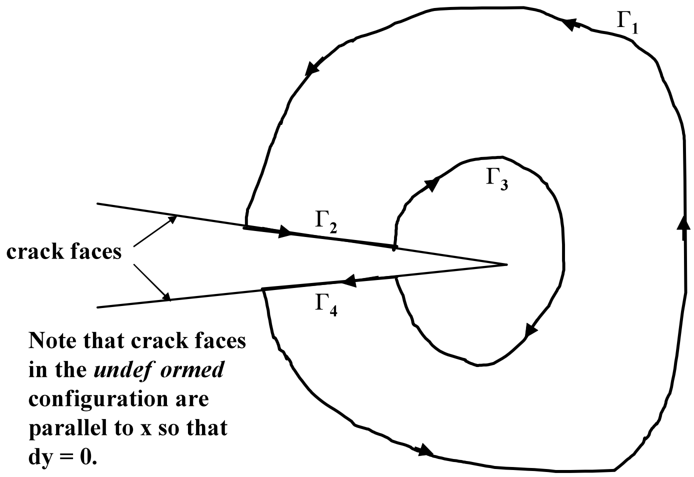
and, referring to the integrand of Equation A3.3, since dy and T_i vanish on the crack surface,

$$J_2 = J_4 = 0 \qquad \text{A3.17}$$

so that Equation A3.16 becomes,

$$J_1 = -J_3 \tag{A3.18}$$

As both Γ_1 and Γ_3 are legitimate Γ contours and the sense around G_3 is clockwise, J_1 and $-J_3$ are legitimate values of the J integral and Equation A3.18 shows that the J integral is path independent.



It is interesting to note that, since J_2 and J_4 vanish and $J_1 = -J_3$, the contribution to the J integral comes entirely from the strain field at the crack tip

4. EQUIVALENCE OF THE J INTEGRAL AND THE ENERGY RELEASE RATE

In order to show the desired equivalence, a body containing a single crack of length λ is subjected to a restricted set of boundary conditions is investigated. Each region of the boundary has one of the two following boundary conditions.

1. The surface traction, T_i, is specified and nonsingular. T_i is defined in Equation A3.4 .
2. The displacement is zero.

For these conditions the potential energy, PE, for this body under the imposed boundary conditions and material restrictions required for J integral validity is,

$$PE = \int_{A_B} W \cdot dx \cdot dy - \int_{\Gamma_B} \sum_i T_i \cdot u_i \cdot ds - \mathbf{G} \cdot t \cdot \lambda \tag{A4.1}$$

where the contour Γ_B must include all of the material in the body. It is taken as the outer boundary of the body not including the crack. This contour is a legitimate J integral contour. Consequently, the area A_B is the area occupied by the body as required for the potential energy. In Equation A4.1,

W = $W(\varepsilon_{ij})$ = strain energy function

ε_{ij} = component of the strain tensor

$$= \tfrac{1}{2} \cdot \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \tag{A4.2}$$

u_i = i^{th} displacement component, assumed to be twice differentiable with respect to x_i
x_i = i^{th} Cartesian coordinate
σ_{ij} = component of the symmetric stress tensor

$$\equiv \frac{\partial W}{\partial \varepsilon_{ij}} \tag{A4.3}$$

ds = differential arc length along Γ_B

Recall, from Equations A1.3, A1.4 and A1.5 that,

$$\frac{\partial PE}{\partial \lambda} = -\,(\text{energy release rate}) \tag{A4.4}$$

Use the definition of the potential energy, Equation A4.1, in the left hand side of the above equation to obtain,

$$\frac{\partial PE}{\partial \lambda} = \frac{\partial}{\partial \lambda}\left[\int_{A_B} W \cdot dx \cdot dy - \int_{\Gamma_B} \sum_i T_i \cdot u_i \cdot ds\right] \tag{A4.5}$$

The derivative on the right hand side in the above equation requires some explanation as the strain energy, W, is computed using a coordinate system located at the tip of the crack with the x direction parallel to the crack. As the crack progresses, the coordinate system must be moved so that $\frac{\partial x}{\partial \lambda} = -1$ and therefore,

$$\frac{\partial}{\partial \lambda} \rightarrow \frac{\partial}{\partial \lambda} + \frac{\partial x}{\partial \lambda} \cdot \frac{\partial}{\partial x} \rightarrow \frac{\partial}{\partial \lambda} - \frac{\partial}{\partial x} \tag{A4.6}$$

That is, there are two contributions to the "partial" derivative. The first term accounts for the change in the stress distribution when the crack grows. The second term accounts for the shift in the position of the coordinate system. Consequently,

$$\frac{\partial}{\partial \lambda} \int_{A_B} W \cdot dx \cdot dy = \int_{A_B} \frac{\partial W}{\partial \lambda} \cdot dx \cdot dy - \int_{\Gamma_B} W \cdot dy \tag{A4.7}$$

$$\frac{\partial}{\partial \lambda} \int_{\Gamma_B} \sum_i T_i \cdot u_i \cdot ds = \int_{\Gamma_B} \sum_i T_i \cdot \frac{\partial u_i}{\partial \lambda} \cdot ds - \int_{\Gamma_B} \sum_i T_i \cdot \frac{\partial u_i}{\partial x} \cdot ds \tag{A4.8}$$

so that Equation A4.7 becomes,

$$\frac{\partial PE}{\partial \lambda} = \int_{A_B} \frac{\partial W}{\partial \lambda} \cdot dx \cdot dy - \int_{\Gamma_B} W \cdot dy - \int_{\Gamma_B} \sum_i T_i \cdot \frac{\partial u_i}{\partial \lambda} \cdot ds + \int_{\Gamma_B} \sum_i T_i \cdot \frac{\partial u_i}{\partial x} \cdot ds \tag{A4.9}$$

According to the Principle of Virtual Work, the difference of the first and third integrals on the right-hand side in the above equation vanish with the result that,

$$\frac{\partial PE}{\partial \lambda} = -\int_{\Gamma_B} W \cdot dy + \int_{\Gamma_B} \sum_i T_i \cdot \frac{\partial u_i}{\partial x} \cdot ds \qquad \text{A4.10}$$

Combining Equations A3.3, A4.4 and A4.10 gives,

$$J = -\frac{\partial PE}{\partial \lambda} = \text{(energy release rate)} \qquad \text{A4.11}$$

5. RELATION BETWEEN CTOD AND THE J INTEGRAL FOR THE DUGDALE-BARENBLATT APPROXIMATION

In each of the solutions presented in the body of this report the crack tip opening displacement, CTOD, is derived and then multiplied by the yield stress, σ_y, to determine the value of the J integral. This section presents the justification for this procedure. In each case the length of the elastic-plastic crack, a, is found. The associated elastic crack has a length L where λ = L-a > 0 and the normal stress across the elastic crack between a and L is σ_y. The length λ is the approximation to the length of the plastic region of the elastic-plastic crack. The J integral from Equation A3.3 is,

$$J = \int_{\Gamma} \left(W \cdot dy - \sum_i T_i \cdot \frac{\partial u_i}{\partial x} \cdot ds \right) \qquad \text{A5.1}$$

where Γ is a contour in the material from one face of the fracture to the other face that encloses the crack tip. For the problems considered here the most convenient contour is a rectangular region enclosing the elastic crack tip. The y-direction height of the rectangle is chosen so small that the first term of the integrand is negligible compared to the second term. The rectangle extends from the elastic-plastic crack tip to the elastic crack tip (crack tip inside rectangle). Note that u_y is zero on the crack axis outside the elastic crack so that,

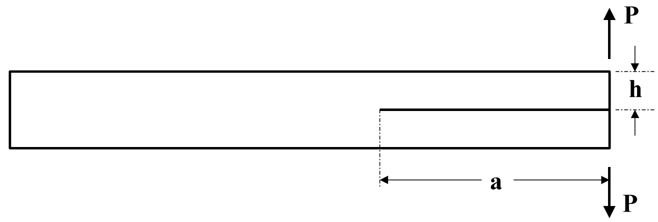
$$J = \int_{x=L}^{x=a} \sigma_y \cdot \left(\left.\frac{\partial u_y}{\partial x}\right|_{\substack{y>0\,\text{side}\\ \text{of crack}}} - \left.\frac{\partial u_y}{\partial x}\right|_{\substack{y<0\,\text{side}\\ \text{of crack}}} \right) \cdot dx = \sigma_y \cdot \left(u_y(a)\Big|_{\substack{y>0\,\text{side}\\ \text{of crack}}} - u_y(a)\Big|_{\substack{y<0\,\text{side}\\ \text{of crack}}} \right) = \sigma_y \cdot \text{CTOD} \qquad \text{A5.2}$$

It is noteworthy that Equation A5.2 has been shown to apply only to analyses that use the Dugdale-Barenblatt approximation. Equation A5.2 is *not* a general result.

APPENDIX 2, A DUGDALE-BARENBLATT TYPE APPROXIMATION FOR THE DOUBLE CANTILEVER BEAM SPECIMEN

INTRODUCTION

DOUBLE CANTILEVER BEAM SPECIMEN



P = SPREADING LOADS
t = SPECIMEN THICKNESS
2·h = SPECIMEN HEIGHT
a = CRACK LENGTH

The sketch above shows the familiar double cantilever beam quasi-static test specimen with monotonically increasing spreading loads applied while the crack propagates. For this analysis the stress state is assumed to be plane stress. The specimen is used to measure static fracture properties of the specimen material. For a brittle material the measured relation between the spreading load, P, and the crack length, a, may be used to determine the elastic energy release rate Γ_{ELAS} when the crack propagates. When elementary beam theory is used for the cantilevers, the elastic energy release rate is predicted to be,

$$\Gamma_{ELAS} = \frac{12 \cdot P^2 \cdot a^2}{t^2 \cdot h^3 \cdot E} \qquad 1$$

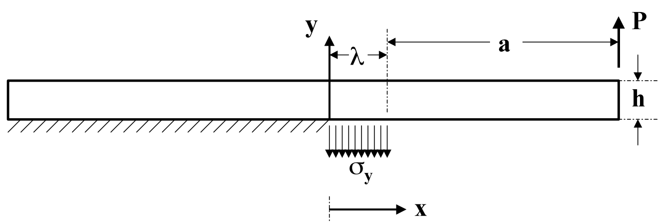
where E is Young's modulus.

In this document an approximation is obtained to take into account the influence of plasticity on the double cantilever beam test. The approximation follows the method used in the Dugdale-Barenblatt plane stress solution for a crack in an infinite medium. That is, a stress singularity is "avoided" by replacing the stresses normal to the crack and in the vicinity of the crack tip with the yield point stress. The size of the region is chosen so that there is no singularity in the stress field.

ANALYSIS WITHOUT SHEAR DEFLECTION OR BASE ROTATION

The sketch below gives the approximation used here. The centerline of the specimen is y = 0. When x < 0 the centerline has no deflection in the y-direction. The region $0 < x < \lambda$ has the yield stress applied to the centerline and the lateral deflection of the bottom surface is not restricted. The region $\lambda < x < \lambda + a$ is modeled as an elementary cantilever beam with a lateral end load, P. There are no side grooves in the specimen.

TOP HALF OF SPECIMEN



x = DISTANCE FROM LHS OF PLASTIC REGION
y = DEFLECTION OF SPECIMEN CENTER LINE
λ = LENGTH OF PLASTIC REGION
σ_y = YIELD STRESS

In the Dugdale-Barenblatt solution the stress intensity factors at the elastic crack tip (x = 0) of a single crack in a two-dimensional infinite medium are determined and the length, λ, of the plastic region is chosen so that at the tip the net stress intensity factor is zero. In this analysis a similar procedure is followed. Consider an elastic crack of length a+λ. In the vicinity of the ends of the elastic crack apply a normal stress to the crack surface. The magnitude of the normal stress is set equal to the yield point, σ_y. This stress represents the load on the elastic region by the plastic region. Consequently, the elastic-plastic crack length (the actual crack length) is equal to a. At the elastic crack tip (x = 0) for the top half of the specimen the stress state, from elementary beam theory, is a linearly varying bending stress in the y-direction and a shear stress with an average value of $\frac{P - \sigma_y \cdot \lambda \cdot t}{h \cdot t}$. The lower half of the specimen has bending stresses and an average shear stress equal to the negative of the average shear stress on the top half of the specimen. If the spreading force is positive at the elastic crack tip, it will develop a stress singularity at the tip. This singularity is proportional to $\sqrt{\Gamma_{ELAS}}$ for a crack of length of λ+a. This singularity is removed by the normal stress equal to the yield point acting in the vicinity ($0 < x < \lambda$) of the tip of the elastic crack. Recall that the length of the elastic-plastic crack is a.

In order to find λ, the magnitude of the J integrals (J integral = calculated energy release rate), Γ_P and Γ_{σ} associated with the P and σ_y loadings are found. Let EI ($= \frac{1}{12} \cdot E \cdot t \cdot h^3$) be the bending stiffness of each beam so that for the P loading (see above sketch),

$$EI \cdot y_P'' = (\lambda + a - x) \cdot P$$
$$EI \cdot y_P' = \left[(\lambda + a) \cdot x - \tfrac{1}{2} \cdot x^2\right] P$$
$$EI \cdot y_P = \left[\tfrac{1}{2} \cdot (\lambda + a) \cdot x^2 - \tfrac{1}{6} \cdot x^3\right] P \qquad , 0 < x < \lambda + a \qquad 2$$
$$U = \tfrac{1}{2} \cdot P \cdot 2 \cdot y_P\big|_{x=\lambda+a} = \tfrac{1}{3} \cdot \frac{P^2 \cdot (\lambda + a)^3}{EI} = \frac{4 \cdot P^2 \cdot (\lambda + a)^3}{E \cdot t \cdot h^3}$$
$$\Gamma_P = \frac{1}{t} \cdot \frac{\partial U}{\partial \lambda} = \frac{12 \cdot P^2 \cdot (\lambda + a)^2}{E \cdot t^2 \cdot h^3}$$

For the σ_y loading,

$$EI \cdot y_\sigma'' = \tfrac{1}{2} \cdot \sigma_y \cdot t \cdot \left(\lambda^2 - 2 \cdot \lambda \cdot x + x^2\right)$$
$$EI \cdot y_\sigma' = \tfrac{1}{2} \cdot \sigma_y \cdot t \cdot \left(\lambda^2 \cdot x - \lambda \cdot x^2 + \tfrac{1}{3} \cdot x^3\right)$$
$$EI \cdot y_\sigma = \tfrac{1}{2} \cdot \sigma_y \cdot t \cdot \left(\tfrac{1}{2} \cdot \lambda^2 \cdot x^2 - \tfrac{1}{3} \cdot \lambda \cdot x^3 + \tfrac{1}{12} \cdot x^4\right) \qquad , 0 < x < \lambda \qquad 3$$
$$U = \tfrac{1}{2} \cdot \sigma_y \cdot t \cdot \int_{x=0}^{x=\lambda} 2 \cdot y_\sigma \cdot dx = \tfrac{1}{20} \cdot \frac{\sigma_y^2 \cdot t^2 \cdot \lambda^5}{EI} = \tfrac{12}{20} \cdot \frac{\sigma_y^2 \cdot t \cdot \lambda^5}{E \cdot h^3}$$
$$\Gamma_\sigma = \frac{1}{t} \cdot \frac{\partial U}{\partial \lambda} = \frac{3 \cdot \sigma_y^2 \cdot \lambda^4}{E \cdot h^3}$$

The condition that $\sqrt{\Gamma_P} = \sqrt{\Gamma_\sigma}$ leads to the equation,

$$\frac{\sigma_y \cdot t \cdot a}{2 \cdot P} \cdot \left(\frac{\lambda}{a}\right)^2 - \frac{\lambda}{a} - 1 = 0 \qquad 4$$

whose positive root is given by,

$$\frac{\lambda}{a} = \frac{P}{\sigma_y \cdot t \cdot a} \cdot \left(1 + \sqrt{1 + 2 \cdot \frac{\sigma_y \cdot t \cdot a}{P}}\right) \qquad 5$$

A result from fracture mechanics that is employed now is that the J integral for this elastic-plastic problem, Γ_{EP}, equals the product of σ_y and the crack tip opening displacement (CTOD). The location of the crack tip for the elastic-plastic problem is at $x = \lambda$ so that,

$$CTOD = 2 \cdot \left(y_P\big|_{x=\lambda} - y_\sigma\big|_{x=\lambda}\right) = \frac{24 \cdot \left[P \cdot \left(\tfrac{1}{2} \cdot (\lambda + a) \cdot \lambda^2 - \tfrac{1}{6} \cdot \lambda^3\right) - \tfrac{1}{2} \cdot \sigma_y \cdot t \cdot \left(\tfrac{1}{4} \cdot \lambda^4\right)\right]}{E \cdot t \cdot h^3} \qquad 6$$

and,

$$\Gamma_{EP} = \frac{12 \cdot \sigma_y^2 \cdot a^4}{E \cdot h^3} \cdot \left[\frac{P}{\sigma_y \cdot t \cdot a} \cdot \left(\left(\frac{\lambda}{a} + 1\right) \cdot \left(\frac{\lambda}{a}\right)^2 - \tfrac{1}{3} \cdot \left(\frac{\lambda}{a}\right)^3\right) - \tfrac{1}{4} \cdot \left(\frac{\lambda}{a}\right)^4\right] \qquad 7$$

so that,

$$\frac{1}{Kr^2} \equiv \frac{\Gamma_{EP}}{\overline{\Gamma}_{ELAS}} = \left(\frac{\sigma_y \cdot t \cdot a}{P}\right)^2 \cdot \left[\frac{P}{\sigma_y \cdot t \cdot a} \cdot \left(\left(\frac{\lambda}{a}+1\right)\cdot\left(\frac{\lambda}{a}\right)^2 - \tfrac{1}{3}\cdot\left(\frac{\lambda}{a}\right)^3\right) - \tfrac{1}{4}\cdot\left(\frac{\lambda}{a}\right)^4\right] \qquad 8$$

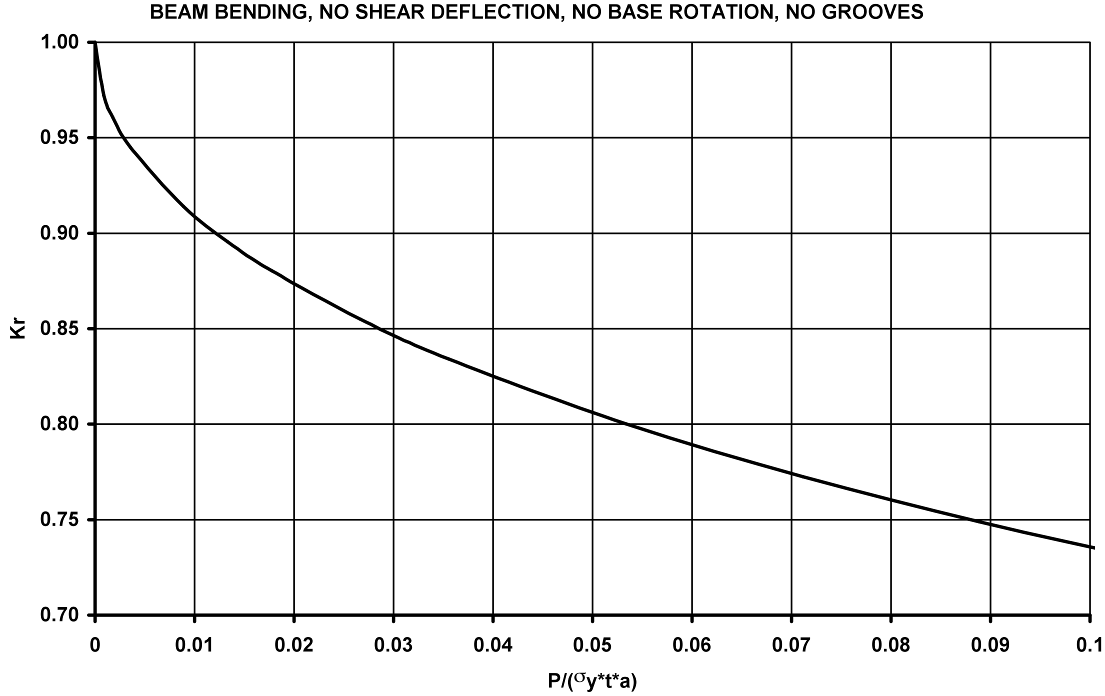
where $\overline{\Gamma}_{ELAS}$ is calculated, using Equation 1, for a crack length of a. The last equation defines Kr^2 which is the approximation for the correction of the elastic energy release rate owing to plasticity influences. If this correction is valid then the fracture propagation energy release rate condition, Γ_{MAT}, is that $\Gamma_{EP} = \Gamma_{MAT}$. This reasoning was used with the Dugdale-Barenblatt solution for the original Failure Assessment Diagram (FAD). This diagram plots Kr as ordinate represented as,

$$Kr = \sqrt{\frac{\overline{\Gamma}_{ELAS}}{\Gamma_{MAT}}} \qquad 9$$

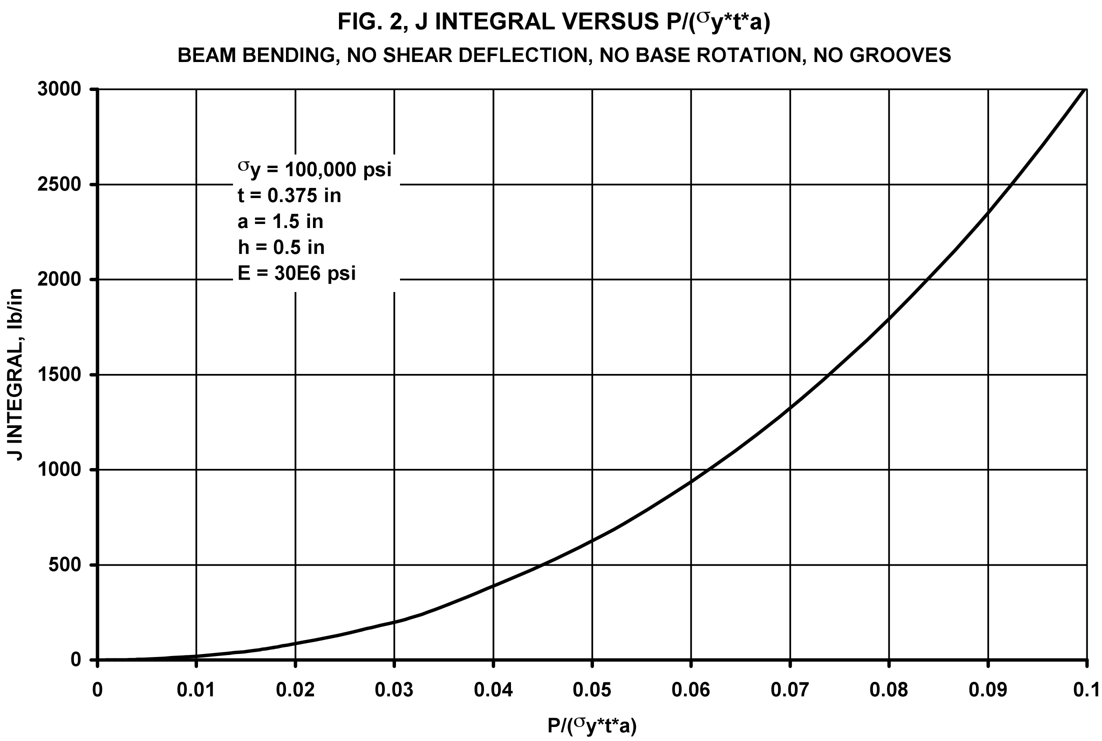
against a measure of the load-to-plastic limit load. Examination of Equations 5 and 8 shows that the abscissa should be $\dfrac{P}{\sigma_y \cdot a \cdot t}$. The figure below shows the FAD curve for this double cantilever beam problem up to $\dfrac{P}{\sigma_y \cdot a \cdot t} = 0.1$ which exceeds the normal values in practice.

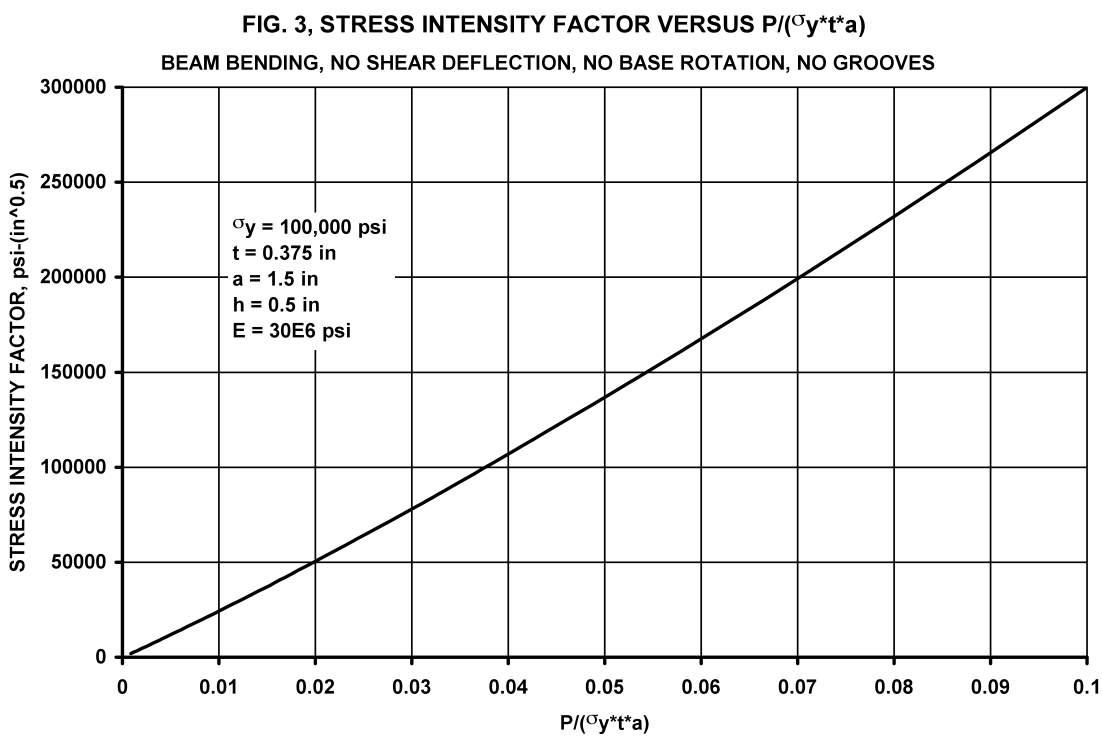
FIG. 1, FAD CURVE FOR THE DCB TEST

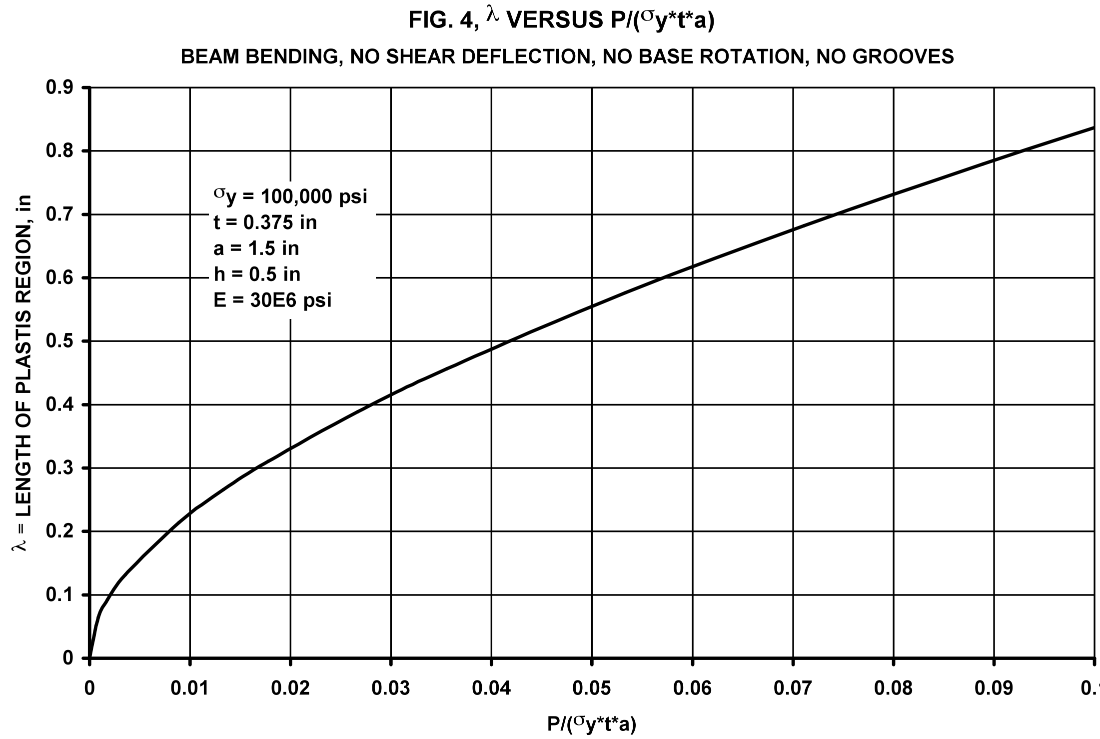
BEAM BENDING, NO SHEAR DEFLECTION, NO BASE ROTATION, NO GROOVES



The three following curves give results based on a typical specimen size and material.







It is interesting to note that, while the beam bending moment at the tip of the elastic-plastic crack equals $P \cdot a$, the beam bending moment at the tip of the elastic crack is zero.

APPENDIX 3, REVIEW OF THE MULTI-YIELD SURFACE, KINEMATIC HARDENING PLASTICITY THEORY

1. GENERAL THEORY

1A. The π-Plane and a Single Yield Surface

The theoretical basis of the plasticity theory used for the PLASTIC program is described in the paper “A Small-Strain Plasticity Theory for Planar Slip Materials” which is appended to this review. In the appended paper a continuum of yield surfaces is used. This continuum, for example, makes it possible to match the results of a tension test exactly. For this study the continuum of yield surfaces is replaced by a discrete number, ISURF, of yield surfaces. This modification produces a tension test stress-strain curve which is piece-wise linear with ISURF+1 linear sections. The description below assumes the reader is somewhat familiar with kinematic hardening.

Consider first an elastic-plastic material with a single yield surface. The yield surface is defined in a strain space with components e_{ij} (i = x, y, z and j = x, y, z). In the annealed, initial state the yield surface is a circular cylinder whose axis passes through the origin in principal strain space. The axis of this cylinder is perpendicular to the π-plane (any plane making equal angles with each of the three principal strain axes). For a specified principal strain state, e_{ij}, the corresponding projection point on the π-plane is not influenced by the addition of a strain increment that is perpendicular to the π-plane. When these perpendicular increments are added to e_{ij} the result, et_{ij} , can be expressed as,

$$et_{ij} = e_{ij} + C \cdot \delta_{ij}$$

where δ_{ij} is the Kronecker delta and is defined by,

$$\delta_{ij} = 1 \quad i = j$$
$$= 0 \quad \text{otherwise}$$

Consequently, the projection on the π-plane of any strain state can be shifted to one, common π-plane. The π-plane containing the origin of the principal strain space is selected here so that the, so called, deviatoric strain state, $\tilde{e}_{ij}$, which plots onto the π-plane containing the origin, is deduced from e_{ij} as,

$$\tilde{e}_{ij} = e_{ij} - \left(\frac{e_{kk}}{3}\right) \cdot \delta_{ij}$$

where repeated i, j or k indices in a term imply summation on x, y, z and $\tilde{e}_{ij}$ obviously has the property,

$$\tilde{e}_{kk} = 0$$

Another way of understanding the projections in the principal strain, e_1, e_2, e_3, space is to use unit vectors $\vec{i}, \vec{j}, \vec{k}$ in the e_1, e_2, e_3 directions, respectively, so that the principal strain vector, $\vec{e}$, becomes,

$$\vec{e} = \vec{i} \cdot e_1 + \vec{j} \cdot e_2 + \vec{k} \cdot e_3$$

This vector can be resolved into components perpendicular to and parallel to the π-plane. The component perpendicular to the π-plane is parallel to the unit vector $\vec{u}$ where,

$$\vec{u} = \frac{1}{\sqrt{3}} \cdot \left(\vec{i} + \vec{j} + \vec{k}\right)$$

The component of $\vec{e}$ in the $\vec{u}$ direction is $\vec{e} \bullet \vec{u}$ so that,

$$\vec{e} \bullet \vec{u} = \frac{1}{\sqrt{3}} \cdot \left(e_1 + e_2 + e_3\right)$$

Then the component of $\vec{e}$ in the π-plane is $\vec{e}$ - $(\vec{e} \bullet \vec{u}) \cdot \vec{u}$ which will be designated $\vec{\tilde{e}}$ so that,

$$\vec{\tilde{e}} \equiv \vec{i} \cdot \tilde{e}_1 + \vec{j} \cdot \tilde{e}_2 + \vec{k} \cdot \tilde{e}_3 = \vec{i} \cdot \left(e_1 - \tfrac{1}{3} \cdot \left(e_1 + e_2 + e_3\right)\right) + \vec{j} \cdot \left(e_2 - \tfrac{1}{3} \cdot \left(e_1 + e_2 + e_3\right)\right) + \vec{k} \cdot \left(e_3 - \tfrac{1}{3} \cdot \left(e_1 + e_2 + e_3\right)\right)$$

The vector $\vec{\tilde{e}}$ lies in the π-plane containing the origin. The length of this vector is $\sqrt{\vec{\tilde{e}} \bullet \vec{\tilde{e}}}$ or,

$$\sqrt{\vec{\tilde{e}} \bullet \vec{\tilde{e}}} = \tilde{e}_1^{\,2} + \tilde{e}_1^{\,2} + \tilde{e}_1^{\,2} = e_1^{\,2} + e_1^{\,2} + e_1^{\,2} - \tfrac{1}{3} \cdot \left(e_1 + e_2 + e_3\right)^2$$

This result may be written in general x, y, z coordinates (remembering that repeated i, j, or k indices in a single term implies summation over that index) as,

$$\sqrt{\vec{\tilde{e}} \bullet \vec{\tilde{e}}} = \sqrt{\tilde{e}_{ij} \cdot \tilde{e}_{ij}} = \sqrt{e_{ij} \cdot e_{ij} - \tfrac{1}{3} \cdot e_{kk}^{\;2}}$$

A single yield surface in the annealed state is a right, circular cylinder of radius $\sqrt{\tfrac{3}{2}} \cdot r$ so that,

$$\tfrac{3}{2} \cdot r^2 = \tilde{e}_{ij} \cdot \tilde{e}_{ij}$$

This seemingly awkward choice for the radius has a historical basis which will not be discussed here. The intersection of this cylinder with the π-plane is obviously a circle in the π-plane.

When plastic flow occurs for this single yield surface case, the circle translates in the π-plane while the yield cylinder axis remains perpendicular to the π-plane. In this process the cylinder axis is displaced from the origin. The strain state principal strains define a strain point in principal strain space. This strain point moves in the strain space as the loads on the material change. So long as the path followed by the strain point lies within the current yield surface further plastic flow will not occur and the material instantaneous response will be elastic.

When the strain point lies on the current yield surface then it is possible for additional plastic flow to occur. Consider a small change of position of the strain point when it lies on the yield surface. If this small change does not take the strain point outside the current yield surface then no additional plastic flow occurs. If the small change takes the strain point outside the current yield surface then the yield surface shifts so that the new strain point lies on the yield surface in its new position. This small change in strain point position causes plastic flow and the current yield circle in the π-plane containing the origin shifts parallel to a line joining the center of the yield circle and the strain point $\vec{\tilde{e}}$ (which also lies in the π-plane containing the origin) on the boundary of the yield circle. The change of position of the center of this yield circle equals the change in plastic strain corresponding to the small change in the position of the strain point. It is worthwhile to note that the flow law described above has the important property that the plastic strain point always lies in the π-plane containing the origin. Therefore the sum of the plastic normal strains vanishes. For this small strain theory the vanishing of this sum is equivalent to the condition that the plastic volume change vanishes. This result also implies that the deviatoric plastic strain equals the plastic strain. The location of the center of the yield circle in the π-plane containing the origin is controlled by the coordinates ρ_1, ρ_2 and ρ_3 of the principal plastic strain vector, $\vec{\rho}$. The corresponding plastic strain coordinates in the general x, y, z coordinate system are ρ_{ij}. As noted above,

$$\rho_{kk} = 0$$

so that,

$$\tilde{\rho}_{ij} = \rho_{ij} - \left(\tfrac{1}{3} \cdot \rho_{kk}\right) \cdot \delta_{ij} = \rho_{ij}$$

In the following presentation $\tilde{\rho}_{ij}$ will be used.

The externally applied stresses, s_{ij}, associated with the material element under consideration are separated into their mean and deviatoric parts as,

$$\text{mean stress} = \tfrac{1}{3} \cdot s_{kk}$$

$$\text{deviatoric stress component} = s_{ij} - \left(\tfrac{1}{3} \cdot s_{kk}\right) \cdot \delta_{ij}$$

The mean stress is related to the volume change of the material. Since there is no plastic volume change the elastic stress-strain equation for volume change is always valid so that,

$$s_{kk} = \frac{E}{1 - 2 \cdot \upsilon} \cdot e_{kk}$$

where

E = Young's modulus of elasticity

υ = Poisson's ratio

The deviatoric stress, $\tilde{s}_{ij}$, is related to the deviatoric elastic strain, $\tilde{e}_{ij} - \tilde{\rho}_{ij}$, through the usual elastic stress-strain law as,

$$\tilde{s}_{ij} = 2 \cdot G \cdot \left(\tilde{e}_{ij} - \tilde{\rho}_{ij}\right)$$

where G, the shear modulus, is given by,

$$G = \frac{E}{2 \cdot (1 + \upsilon)}$$

The condition for yielding in this plasticity theory is influenced by the existence of a back stress. This back stress is hypothesized on the basis of dislocation pile-ups at the grain boundaries of the polycrystalline metals under consideration. For the case where the dislocations cannot cross the grain boundaries it is reasonable to assume the back stress is proportional to the plastic strain. Therefore, define the back stress to be $\lambda \cdot \tilde{\rho}_{ij}$ and then the condition for yielding should be related to the difference between the deviatoric stress $2 \cdot G \cdot \left(\tilde{e}_{ij} - \tilde{\rho}_{ij}\right)$ and the back stress $\lambda \cdot \tilde{\rho}_{ij}$. For the yield condition the strain quantity of interest is $\tilde{e}_{ij} - \kappa \cdot \tilde{\rho}_{ij}$ where,

$$\kappa = 1 + \frac{\lambda}{2 \cdot G}$$

Thus the yield condition after plastic flow has occurred is,

$$\left(\tilde{e}_{ij} - \kappa \cdot \tilde{\rho}_{ij}\right) \cdot \left(\tilde{e}_{ij} - \kappa \cdot \tilde{\rho}_{ij}\right) = \tfrac{3}{2} \cdot r^2$$

which defines the yield condition on the π-plane containing the origin as a circle with radius $\sqrt{\tfrac{3}{2}} \cdot r$ and center coordinates $\kappa \cdot \rho_1$, $\kappa \cdot \rho_2$, $\kappa \cdot \rho_3$. When the above yield condition is satisfied and the next small increment of strain, $\Delta\tilde{e}_{ij}$, is such that,

$$\left(\tilde{e}_{ij} - \kappa \cdot \tilde{\rho}_{ij}\right) \cdot \Delta\tilde{e}_{ij} > 0$$

then plastic flow occurs. The corresponding change in the plastic strain, $\Delta\tilde{\rho}_{ij}$, is determined by the requirement that the new strain points defined by $\tilde{e}_{ij} + \Delta\tilde{e}_{ij}$ and $\tilde{\rho}_{ij} + \Delta\tilde{\rho}_{ij}$ satisfy the yield condition. This requirement, to first order changes in the strains, gives,

$$\kappa \cdot \Delta\tilde{\rho}_{ij} = \frac{2}{3 \cdot r^2} \cdot \left(\tilde{e}_{ij} - \kappa \cdot \tilde{\rho}_{ij}\right) \cdot \left(\tilde{e}_{km} - \kappa \cdot \tilde{\rho}_{km}\right) \cdot \Delta\tilde{e}_{km}$$

which is a shift of the yield circle center in the π-plane containing the origin in the $\vec{\tilde{e}} - \kappa \cdot \vec{\tilde{\rho}}$ direction.

1B. Extension from a Single Yield Surface to Multiple Yield Surfaces

The appended paper uses a continuum of yield surfaces to represent the material model. In the formulation given here a discrete number, ISURF, of yield surfaces, replaces the continuum. A physical basis for thinking of these yield surfaces is that each surface represents a fraction of the crystals in the polycrystalline material. This fraction would include all crystals with orientations whose yield stresses approximate the yield stress value for that yield surface. The fundamental notions are that each crystal has the same total strain, e_{ij} , and that the stresses vary from crystal to crystal according to the orientation of the crystal.

The stress state is given as a summation over all the crystals according to their fractional representation. Since there is no volumetric plastic strain the volumetric elastic law,

$$s_{kk} = \frac{E}{1 - 2 \cdot \upsilon} \cdot e_{kk}$$

applies in this case. The deviatoric stress is given by,

$$\tilde{s}_{ij} = \sum_{l=1}^{ISURF} K(l) \cdot \left(\tilde{e}_{ij} - \tilde{\rho}_{ij}(l) \right)$$

The quantity K(l) is an elastic constant corresponding to the l^{th} yield surface and $\tilde{\rho}_{ij}(l)$ is the plastic strain associated with the l^{th} yield surface. Since $\tilde{\rho}_{ij}(l)$ will equal zero for all values of l in the annealed state and e_{ij} is measured from the annealed, stressless configuration, the initial response is elastic and the K(l) values must satisfy the condition,

$$\sum_{l=1}^{ISURF} K(l) = 2 \cdot G = \frac{E}{1 + \upsilon}$$

so that,

$$\tilde{s}_{ij} = \frac{E}{1 + \upsilon} \cdot \tilde{e}_{ij} - \sum_{l=1}^{ISURF} K(l) \cdot \tilde{\rho}_{ij}(l)$$

and

$$\begin{aligned} s_{ij} &= \frac{E}{3 \cdot (1 - 2 \cdot \nu)} \cdot e_{kk} \cdot \delta_{ij} + \frac{E}{1 + \upsilon} \cdot \tilde{e}_{ij} - \sum_{l=1}^{ISURF} K(l) \cdot \tilde{\rho}_{ij}(l) \\ &= \frac{\nu \cdot E}{(1 - 2 \cdot \nu) \cdot (1 + \upsilon)} \cdot e_{kk} \cdot \delta_{ij} + \frac{E}{1 + \upsilon} \cdot e_{ij} - \sum_{l=1}^{ISURF} K(l) \cdot \tilde{\rho}_{ij}(l) \end{aligned}$$

The yield condition for the l^{th} yield surface is,

$$\left(\tilde{e}_{ij} - \kappa \cdot \tilde{\rho}_{ij}(l)\right) \cdot \left(\tilde{e}_{ij} - \kappa \cdot \tilde{\rho}_{ij}(l)\right) = \tfrac{3}{2} \cdot r(l)^2$$

where the values of r(l) are ordered so that r(k+1) > r(k). Note that the same value of κ is used for all the yield conditions so that the back stress for the l^{th} yield surface has a proportionality to $\tilde{\rho}_{ij}(l)$ that is independent of l.

The condition that the yield condition is continuously satisfied during loading is found by requiring that when the small change of strain $\tilde{e}_{ij} \rightarrow \tilde{e}_{ij} + \Delta\tilde{e}_{ij}$ occurs the corresponding change in $\tilde{\rho}_{ij} \rightarrow \tilde{\rho}_{ij} + \Delta\tilde{\rho}_{ij}$ be governed by,

$$\left(\tilde{e}_{ij} - \kappa \cdot \tilde{\rho}_{ij}(l)\right) \cdot \left(\Delta\tilde{e}_{ij} - \kappa \cdot \Delta\tilde{\rho}_{ij}(l)\right) = 0$$

so that,

$$\Delta\tilde{\rho}_{ij}(I) = \frac{2}{3} \cdot \frac{\left(\tilde{e}_{ij} - \kappa \cdot \tilde{\rho}_{ij}(I)\right)}{\tilde{r}(I)^2} \cdot \left(\tilde{e}_{mn} - \kappa \cdot \tilde{\rho}_{mn}(I)\right) \cdot \Delta\tilde{e}_{mn}$$

This completes the general theoretical basis. The required constants for the complete model of the polycrystalline material are,

υ = Poisson's ratio
κ = work hardening parameter
ISURF = number of yield surfaces
K(l) = modified stiffness, l = 1, ISURF
r(l) = $\sqrt{\tfrac{2}{3}} \cdot$ (l^{th} yield surface radius in the π-plane), l = 1, ISURF

These values must be obtained from experiments.

2. USING THE TENSION TEST TO EVALUATE THE MATERIAL PROPERTIES

When Poisson's ratio and a tensile test stress-strain curve for an annealed specimen are known, then the parameters for the material model may be found. The stress-strain curve starts at coordinates (0,0) and extends to the maximum stress of the tension test. The maximum strain in the tensile test must cover the extent of the strain being modeled. As this model is for small strains the maximum strain for the mathematical model should not exceed about 0.20.

Most annealed metals have stress-strain curves, in the region of 0. to 0.2 strain, that begins with an elastic region and ends with a linear section whose slope is much less than the original slope of the material. The first step for finding the model parameters is to select a number, ISURF, of points (not including the origin) on the stress-strain curve that give an adequate representation of the curve. The first point should be at the end of the elastic region and the last point should be at the beginning of the linear part for larger strains. Identify the points chosen on the stress-strain curve as,

$e_{ax}(l)$ $= l^{th}$ axial strain, $e_{ax}(k+1) > e_{ax}(k)$, $l = 1$, ISURF

$s_{ax}(l)$ $= l^{th}$ axial stress corresponding to $e_{ax}(l)$, $l = 1$, ISURF

In addition, the ratio of the slopes of the stress-strain curve for the large strain, linear part to the elastic region should be evaluated. Denote this ratio as [slope ratio].

The value of the Poisson's ratio, υ, is specified (it must be less than 0.5) and Young's modulus, E, is given by,

$$E = \frac{s_{ax}(1)}{e_{ax}(1)}$$

then the shear modulus, G, is found from,

$$G = \frac{E}{2\cdot(1 + \upsilon)}$$

In order to deduce further properties for the model from the stress-strain data it is necessary to solve the tension test problem using the mathematical model equations described above in the preceding section. Let the z-axis lie in the direction of the tensile loading stress, s_{ax} . Then s_{zz} equals s_{ax} and all other stresses vanish so that the non-vanishing deviatoric stress components are,

$$\tilde{s}_{xx} = -\tfrac{1}{3}\cdot s_{ax}$$

$$\tilde{s}_{yy} = -\tfrac{1}{3}\cdot s_{ax}$$

$$\tilde{s}_{zz} = +\tfrac{2}{3}\cdot s_{ax}$$

Owing to the symmetry about the z axis the axial, e_{ax}, and lateral, - e_{lat}, strains are the only strains so that the non-vanishing strain components are,

$$e_{xx} = - e_{lat}$$

$$e_{yy} = - e_{lat}$$

$$e_{zz} = e_{ax}$$

and the only non-vanishing deviatoric strains are,

$$\tilde{e}_{xx} = -\tfrac{1}{3}\cdot\left(e_{ax} + e_{lat}\right)$$

$$\tilde{e}_{yy} = -\tfrac{1}{3}\cdot\left(e_{ax} + e_{lat}\right)$$

$$\tilde{e}_{zz} = \tfrac{2}{3}\cdot\left(e_{ax} + e_{lat}\right)$$

The symmetry of the problem also leads to a simplification of the plastic strain formulation. The non-vanishing components of the plastic strains may be written for l from 1 to ISURF as,

$$\tilde{\rho}_{xx}(l) = \rho_{xx}(l) = -\tfrac{1}{2}\cdot\tilde{\rho}_{ax}(l) = -\tfrac{1}{2}\cdot\rho_{ax}(l)$$

$$\tilde{\rho}_{yy}(l) = \rho_{yy}(l) = -\tfrac{1}{2}\cdot\tilde{\rho}_{ax}(l) = -\tfrac{1}{2}\cdot\rho_{ax}(l)$$

$$\tilde{\rho}_{zz}(l) = \rho_{zz}(l) = \tilde{\rho}_{ax}(l) = \rho_{ax}(l)$$

The l^{th} yield condition becomes,

$$r(l) = \tfrac{2}{3}\cdot\left(e_{ax} + e_{lat}\right) - \kappa\cdot\rho_{ax}(l)$$

When this yield condition is satisfied the corresponding loading condition is,

$$\left(\tfrac{2}{3}\cdot\left(e_{ax} + e_{lat}\right) - \kappa\cdot\rho_{ax}(l)\right)\cdot\left(\Delta e_{ax} + \Delta e_{lat}\right) > 0$$

When both the yield condition and the loading condition are satisfied for a particular value of l the incremental plastic flow is given by,

$$\kappa\cdot\Delta\rho_{ax}(l) = \tfrac{2}{3}\cdot\left(\Delta e_{ax} + \Delta e_{lat}\right)$$

The elastic compressibility condition is,

$$s_{ax} = \frac{E}{1 - 2\cdot\upsilon}\cdot\left(e_{ax} - 2\cdot e_{lat}\right)$$

so that e_{lat} may be written as,

$$e_{lat} = \tfrac{1}{2} \cdot e_{ax} - (1 - 2 \cdot \upsilon) \cdot \frac{s_{ax}}{E}$$

The axial deviatoric stress-strain equation gives,

$$\tfrac{2}{3} \cdot s_{ax} = \frac{E}{1 + \upsilon} \cdot \tfrac{2}{3} \cdot (e_{ax} + e_{lat}) - \sum_{l=1}^{ISURF} K(l) \cdot \rho_{ax}(l)$$

This equation may be written in incremental form as,

$$\Delta s_{ax} = \frac{E}{1 + \upsilon} \cdot (\Delta e_{ax} + \Delta e_{lat}) - \sum_{l=1}^{ISURF} K(l) \cdot \Delta\rho_{ax}(l)$$

Before continuing to find the K(l) quantities, consider the case where the strain point has contacted all the yield surfaces and the loading is continued. This case can be used to find κ. In this case,

$$\Delta\rho_{ax}(l) = \tfrac{2}{3} \cdot (\Delta e_{ax} + \Delta e_{lat}) / \kappa \qquad \text{for } l = 1, \cdots, ISURF$$

Combining the last two equations and using the condition that the sum of all the K(I) values equals $\frac{E}{1 + \upsilon}$ gives,

$$\Delta s_{ax} = \frac{E}{1 + \upsilon} (\Delta e_{ax} + \Delta e_{lat}) \cdot \left(1 - \frac{1}{\kappa}\right)$$

The equation for e_{lat} given above (i.e.-five equations above) may be written in incremental form as,

$$\Delta e_{lat} = \tfrac{1}{2} \cdot \Delta e_{ax} - (1 - 2 \cdot \upsilon) \cdot \frac{\Delta s_{ax}}{E}$$

Eliminating Δe_{lat} between the last two equations and noting that the quantity [slope ratio] defined earlier is given by,

$$[\text{slope ratio}] = \frac{\Delta s_{ax}}{E \cdot \Delta e_{ax}}$$

yields,

$$\frac{1}{\kappa} = 1 - \frac{(1 + \upsilon) \cdot [\text{slope ratio}]}{\tfrac{3}{2} - (1 - 2\upsilon) \cdot [\text{slope ratio}]}$$

so that this equation determines κ based on data from the measured tensile test stress-strain curve.

Now return to the deviatoric stress-strain equation and the incremental form of the equation for e_{lat}. Combining these equations to eliminate Δe_{lat} gives,

$$\frac{2 - \upsilon}{1 + \upsilon} \cdot \Delta s_{ax} = \frac{3 \cdot E}{2 \cdot (1 + \upsilon)} \cdot \Delta e_{ax} - \tfrac{3}{2} \cdot \sum_{l=1}^{ISURF} K(l) \cdot \Delta\rho_{ax}(l)$$

The values for K(l) for l from 1 to ISURF are found successively using the last equation. Referring now to the input data $e_{ax}(l)$, $s_{ax}(l)$, $l = 1, \cdots$, ISURF described previously, let,

$\Delta s_{ax} = s_{ax}(2) - s_{ax}(1)$
$\Delta e_{ax} = e_{ax}(2) - e_{ax}(1)$

For this increment of loading the plastic strains, $\rho_{ax}(l)$, are,

$$\Delta\rho_{ax}(1) = \tfrac{2}{3} \cdot \left(\Delta e_{ax} + \Delta e_{lat}\right) / \kappa = \left(\Delta e_{ax} - \tfrac{2}{3} \cdot (1 - 2 \cdot \upsilon) \cdot \frac{\Delta s_{ax}}{E}\right) / \kappa$$

$$\Delta\rho_{ax}(k) = 0 \qquad \text{for } k = 2, \cdots, \text{ISURF}$$

Then, from the incremental, deviatoric stress-strain equation, the value of K(1) may be found. This equation is,

$$\frac{3 \cdot E}{2 \cdot (1 + \upsilon)} \cdot \Delta e_{ax} - \frac{2 - \upsilon}{1 + \upsilon} \cdot \Delta s_{ax} = \tfrac{3}{2} \cdot K(1) \cdot \frac{\Delta e_{ax} - \tfrac{2}{3} \cdot (1 - 2 \cdot \upsilon) \cdot \frac{\Delta s_{ax}}{E}}{\kappa}$$

When K(1) has been found using this equation, K(2) may be found using,

$\Delta s_{ax} = s_{ax}(3) - s_{ax}(2)$
$\Delta e_{ax} = e_{ax}(3) - e_{ax}(2)$

and,

$$\Delta\rho_{ax}(1) = \Delta\rho_{ax}(2) = \left(\Delta e_{ax} - \tfrac{2}{3} \cdot (1 - 2 \cdot \upsilon) \cdot \frac{\Delta s_{ax}}{E}\right) / \kappa$$

$$\Delta\rho_{ax}(k) = 0 \qquad \text{for } k = 3, \cdots, \text{ISURF}$$

so that

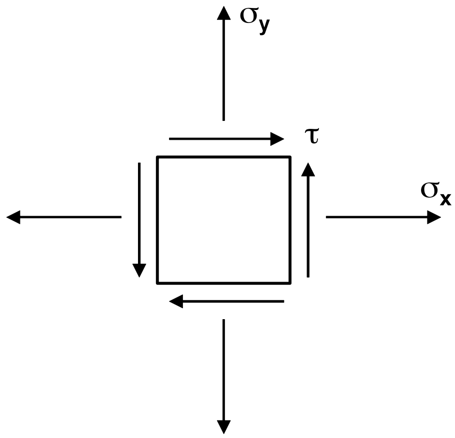
$$\frac{3 \cdot E}{2 \cdot (1 + \upsilon)} \cdot \Delta e_{ax} - \frac{2 - \upsilon}{1 + \upsilon} \cdot \Delta s_{ax} = \tfrac{3}{2} \cdot (K(1) + K(2)) \cdot \frac{\Delta e_{ax} - \tfrac{2}{3} \cdot (1 - 2 \cdot \upsilon) \cdot \dfrac{\Delta s_{ax}}{E}}{\kappa}$$

This procedure can be repeated for each input strain and stress increment so that K(1),$\cdots$, K(ISURF - 1) can be found. Then the condition that the sum of K(l) for l from 1 to ISURF equals $\dfrac{E}{1 + \upsilon}$ may be used to find K(ISURF) as,

$$K(ISURF) = \frac{E}{1 + \upsilon} - \sum_{l=1}^{ISURF - 1} K(l)$$

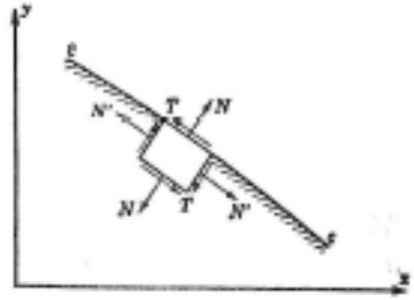
The procedure described above determines all the required parameters for the mathematical model based on a specified value of Poisson's ratio and a stress-strain curve.

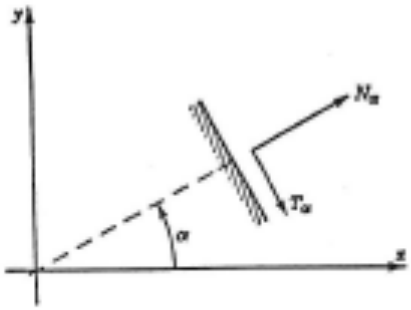
APPENDIX 4, SOME FORMULAS



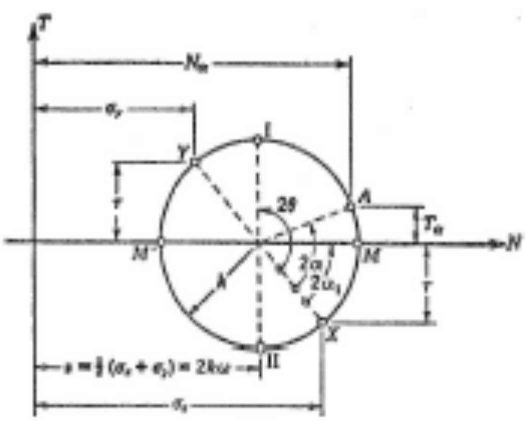
$$\sigma_x = 2 \cdot k \cdot \omega + k \cdot \sin(2 \cdot \theta)$$
$$\sigma_y = 2 \cdot k \cdot \omega - k \cdot \sin(2 \cdot \theta)$$
$$\tau = -k \cdot \cos(2 \cdot \theta)$$

$$\omega = \frac{\sigma_x + \sigma_y}{4 \cdot k} = \frac{\sigma}{2 \cdot k}$$
$$\theta = \tfrac{1}{2} \cdot \arctan\left(\frac{2 \cdot \tau}{\sigma_x - \sigma_y}\right) + \tfrac{\pi}{4}$$





$$N_\alpha = \sigma_x \cdot \cos^2(\alpha) + \sigma_y \cdot \sin^2(\alpha) + 2 \cdot \tau \cdot \cos(\alpha) \cdot \sin(\alpha)$$
$$T_\alpha = (\sigma_x - \sigma_y) \sin(\alpha) \cdot \cos(\alpha) + \tau \cdot \cdot (\sin^2(\alpha) - \cos^2(\alpha))$$



(a)

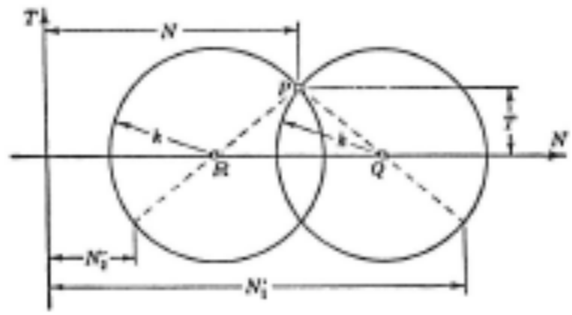
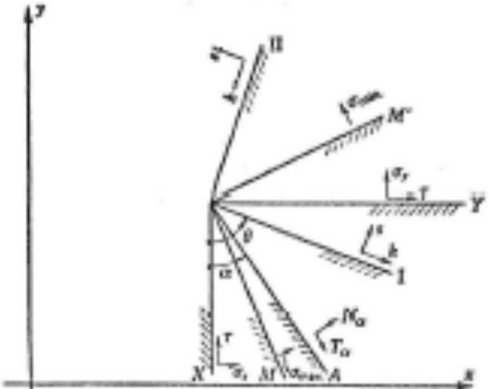


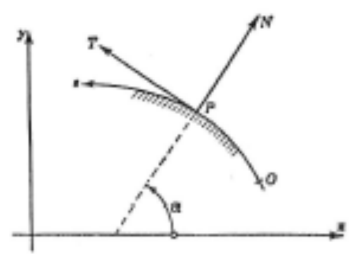
FIG. 39. Mohr circles for given boundary stresses.



(b)

$$N = 2 \cdot k \cdot \omega + k \cdot \sin(2 \cdot (\theta - \alpha))$$
$$T = -k \cdot \cos(2 \cdot (\theta - \alpha))$$
$$N' = 2 \cdot k \cdot \omega - k \cdot \sin(2 \cdot (\theta - \alpha))$$

$$\theta = \alpha \pm \tfrac{1}{2} \cdot \left(\pi - \arccos\left(\frac{T}{k}\right)\right)$$
$$\omega = \tfrac{1}{2} \cdot \left(\frac{N}{k} - \sin(2 \cdot (\theta - \alpha))\right)$$



$$\int_V F_{,i} \cdot dV = \int_S F \cdot n_i \cdot dS$$

$$\int_V \sigma_{ij} \cdot \varepsilon_{ij} \cdot dV = \int_V \left(\sigma_{ij} \cdot u_i\right)_{,j} \cdot dV = \int_S \sigma_{ij} \cdot u_i \cdot n_j \cdot dS = \int_S T_i \cdot u_i \cdot dS$$

$$X = \sigma_x \cdot \cos(\alpha) + \tau \cdot \sin(\alpha)$$
$$Y = \sigma_y \cdot \sin(\alpha) + \tau \cdot \cos(\alpha)$$

Polar Coordinates

$$\frac{\partial \sigma_r}{\partial r} + \frac{1}{r} \cdot \frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{\sigma_r - \sigma_\theta}{r} = 0$$
$$\frac{1}{r} \cdot \frac{\partial \sigma_\theta}{\partial \theta} + \frac{\partial \tau_{r\theta}}{\partial r} + \frac{2 \cdot \tau_{r\theta}}{r} = 0$$

$$\varepsilon_r = \frac{\partial u}{\partial r}$$
$$\varepsilon_\theta = \frac{u}{r} + \frac{\partial v}{r \cdot \partial \theta}$$
$$\gamma_{r\theta} = \frac{\partial u}{r \cdot \partial \theta} + \frac{\partial v}{\partial r} - \frac{v}{r}$$

Schwarz' Inequality

$$\vec{a} = a_x \cdot \vec{i} + a_y \cdot \vec{j} + a_z \cdot \vec{k}, \quad \vec{b} = b_x \cdot \vec{i} + b_y \cdot \vec{j}$$
$$\left(a_x^2 + a_y^2\right)\left(b_x^2 + b_y^2\right) \geq \left(a_x \cdot b_x + a_y \cdot b_y\right)^2$$

The last equation may be verified by expansion

i.e. - the product of the lengths is greater than their dot product squared.

APPENDIX 5, THE PRANDL-REUSS EQUATIONS FOR PERFECTLY FLASTIC FLOW

Stresses: σ_x, σ_y, σ_z, τ_{xy}, τ_{yz}, τ_{zx}

Reduced stresses:

$$s = \tfrac{1}{3} \cdot \left(\sigma_x + \sigma_y + \sigma_z\right)$$
$$s_x = \sigma_x - s$$
$$s_y = \sigma_y - s$$
$$s_z = \sigma_z - s$$

shear stresses unchanged

Engineering strains: ε_x, ε_y, ε_z, γ_{xy}, γ_{yz}, γ_{zx}

Reduced strains:

$$e = \tfrac{1}{3} \cdot \left(\varepsilon_x + \varepsilon_y + \varepsilon_z\right)$$
$$e_x = \varepsilon_x - e$$
$$e_y = \varepsilon_y - e$$
$$e_z = \varepsilon_z - e$$

shear strains unchanged

Hooke's law

$$G = \frac{E}{2 \cdot (1 + \nu)}$$
$$K = \frac{E}{3 \cdot (1 - 2 \cdot \nu)}$$
$$2 \cdot G \cdot \dot{e}_x{}' = \dot{s}_x$$
$$2 \cdot G \cdot \dot{e}_y{}' = \dot{s}_y$$
$$2 \cdot G \cdot \dot{e}_z{}' = \dot{s}_z$$
$$G \cdot \dot{\gamma}_{xy}{}' = \dot{\tau}_{xy}$$
$$G \cdot \dot{\gamma}_{yz}{}' = \dot{\tau}_{yz}$$
$$G \cdot \dot{\gamma}_{zx}{}' = \dot{\tau}_{zx}$$
$$3 \cdot K \cdot \dot{e} = \dot{s}$$

Von Mises yield condition

$$\sigma_x^2 + \sigma_y^2 + \sigma_z^2 - \sigma_x \cdot \sigma_y - \sigma_y \cdot \sigma_z - \sigma_z \cdot \sigma_x + 3 \cdot \left(\tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2\right) = \sigma_{YP}^2 = 3 \cdot k^2$$

Prandtl-Reuss formulation

$e = 0$

$$2 \cdot G \cdot \dot{e}_x = \dot{s}_x + \lambda \cdot s_x$$

$$2 \cdot G \cdot \dot{e}_y = \dot{s}_y + \lambda \cdot s_y$$

$$2 \cdot G \cdot \dot{e}_z = \dot{s}_z + \lambda \cdot s_z$$

$$G \cdot \dot{\gamma}_{xy} = \dot{\tau}_{xy} + \lambda \cdot \tau_{xy}$$

$$G \cdot \dot{\gamma}_{yz} = \dot{\tau}_{yz} + \lambda \cdot \tau_{yz}$$

$$G \cdot \dot{\gamma}_{zx} = \dot{\tau}_{zx} + \lambda \cdot \tau_{zx}$$

$$\dot{W} = s_x \cdot \dot{e}_x + s_y \cdot \dot{e}_y + s_z \cdot \dot{e}_z + \tau_{xy} \cdot \dot{\gamma}_{xy} + \tau_{yz} \cdot \dot{\gamma}_{yz} + \tau_{zx} \cdot \dot{\gamma}_{zx}$$

= rate of distortion work

$$G \cdot \dot{W} = \lambda \cdot k^2$$

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