AN ELEMENTARY, BRIEF REVIEW OF THE METHOD OF MUSKHELISHVILI IN ELASTICITY THEORY

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INTRODUCTION

Many structural engineers and machine designers who are well versed in strength of materials techniques have little opportunity to use potential theory (i.e. – solutions governed by Laplace's equation). This report is an attempt to make one of the applications of potential theory in elasticity more clear. A modest background in complex functions is assumed, including the Cauchy Residue and Integral Theorems. For almost 100 years the two dimensional mathematical theory of elasticity has employed complex variable theory to obtain solutions to certain, specific problems. One application that has developed is for thin infinite plates containing holes of finite extent. Another application is in fracture mechanics for sharp cracks in plates. A technique that addresses such problems is reviewed in this report. This technique is often referred to as the method of Muskhelishvili (Reference 1). Only the problem of a hole in an infinite plate is reviewed here. The stresses at infinity vanish and the loads on the hole boundary are self equilibrating. The coordinate systems defined in Reference 2 are adopted for this report.

This method is based on the biharmonic stress function familiar in two-dimensional elasticity and known as the Airy stress function, Φ. When this function satisfies the biharmonic equation ($\nabla^4\Phi = 0$), it can be used to find a stress field that is a solution to the elasticity governing equations (Reference 2, Chapter 2) as (in Cartesian coordinates),

$$\sigma_x = \frac{\partial^2 \Phi}{\partial y^2}, \quad \sigma_y = \frac{\partial^2 \Phi}{\partial x^2}, \quad \tau_{xy} = -\frac{\partial^2 \Phi}{\partial x\, \partial y}$$

Any function satisfying the biharmonic equation can, in general, be replaced by two complex harmonic functions, $g_1(z)$ and $g_2(z)$, satisfying Laplace's equation ($\nabla^2 g_1(z) = 0,\ \nabla^2 g_2(z) = 0$) with $z = x + i \cdot y$. Then,

$$\Phi = \mathrm{Re}\left[\bar{z} \cdot g_1(z) + g_2(z)\right] = \tfrac{1}{2} \cdot \left(\bar{z} \cdot g_1(z) + g_2(z) + z \cdot \bar{g}_1(\bar{z}) + \bar{g}_2(\bar{z})\right)$$

where Re designates real part and $\bar{z}$ is the conjugate of z ($\bar{z} = x - i \cdot y$). A proof of the above result is given in both References 1 and 2. A general result that is useful here is,

$$\frac{\partial \Phi}{\partial x} + i \cdot \frac{\partial \Phi}{\partial y} = g_1(z) + z \cdot \bar{g}_1{}'(\bar{z}) + \bar{g}_2{}'(\bar{z})$$

The boundary conditions for the two harmonic functions can be formulated in a straightforward way. The next step is to determine a conformal mapping that converts the hole boundary to the unit circle. For each of the two functions the Cauchy residue theorem and other properties of complex functions are used to find the solutions for the functions outside the hole. Once the two functions are found the stresses and displacements can be determined directly.

The primary reference for this method is Reference 1. It is exhaustive and mathematically oriented. An elementary formulation more easily comprehended by most engineers is presented in Reference 2, Chapter 6. Reference 3 presents the required complex variable basis in a presentation that is easy to understand.

The following section and Appendix A of this report give a detailed description for the solution of a thin infinite plate with a hole, vanishing stress at infinity and a self equilibrating loading on the hole boundary. The subsequent section contains an illustrative example of the circular hole. The last section gives, as a second illustrative example, a case of an elliptical hole.

Appendix B contains a FORTRAN source code and illustrative input data file for a program that determines stresses and displacements for the elliptical hole problem considered here.

A special effort has been made in the preparation of this text to make it easy to follow and understand. The result is that many redundancies appear below that are not usually present in contemporary technical literature. In other words, this presentation is intended to be a pedagogical document.

DESCRIPTION OF THE METHOD OF MUSKHELISHVILI

Familiarity with the Cauchy residue theorem is assumed in the following. This theorem for a complex harmonic function f(z) of a complex argument with pole singularities only inside a generic boundary is given by,

$$\oint_C f(z) \cdot dz = 2 \cdot \pi \cdot i \cdot \sum_{k=1}^{n} \operatorname{Res}(a_k)$$

$$\operatorname{Res}(a_k) = \frac{1}{(m_k - 1)\,!} \cdot \left[\frac{d^{m_k - 1}}{dz^{m_k - 1}} \left\{ (z - a_k)^{m_k} \cdot f(z) \right\} \right]_{z=a_k}$$

where a_k is the location in the z-plane of the k^{th} pole and m_k is its order. Recall that (0)! = 1. More precisely, if C is the boundary of a region in which f(z) is analytic except at a finite number of poles, then $\int_C f(z) \cdot dz$ is given by $2 \cdot \pi \cdot i$ times the sum of the residues of f(z) in the region.

A second important result is that if a function f(z) is analytic everywhere outside of a region C then,

$$\oint_C \frac{f(\sigma) \cdot d\sigma}{\sigma - z} = -2 \cdot \pi \cdot i \cdot f(z) \quad \text{z outside of C}$$

where σ defines the boundary for the contour integration.

The problems described here are for thin, plates of infinite extent with single holes (plane stress conditions are assumed). The boundary conditions are that the stresses at infinity vanish and the boundary loads on the hole are self equilibrating (both force and moment). There is no other loading on the plate.

The first step, often the most challenging step, is to find the conformal mapping function that maps the hole boundary in the z-plane into the unit circle in the ζ-plane. The z-plane is the physical plane and the ζ-plane is the mapped plane where,

$$z = x + i \cdot y = r \cdot e^{i \cdot \hat{\alpha}} \qquad \bar{z} = x - i \cdot y = r \cdot e^{-i \cdot \hat{\alpha}} \qquad \hat{\alpha} = \arctan\left(\frac{y}{x}\right)$$

$$\zeta = \xi + i \cdot \eta = \rho \cdot e^{i \cdot \theta} \qquad \bar{\zeta} = \xi - i \cdot \eta = \rho \cdot e^{-i \cdot \theta} \qquad \theta = \arctan\left(\frac{\eta}{\xi}\right)$$

In the mapped plane the hole boundary whose radius equals one is,

$$\zeta\Big|_{\text{boundary}} \equiv \sigma = e^{i \cdot \theta}$$

This mapping from the ζ-plane to the z-plane is expressed as,

$$z = \omega(\zeta)$$

where the boundary of the hole, $z|_{\text{boundary}}$, is given by,

$$z|_{\text{boundary}} = \omega\left(e^{i\cdot\theta}\right)$$

Consider a point E on a line in the z-plane defined by ξ = constant. The angle the normal to this line at E makes with the x-axis for increasing ξ is defined as α. When ω(ζ) is specified, tan(α) may be easily found. Let the function F(x,y) = 0 define the hole boundary in the z-plane. Then,

$$\frac{\partial F(x,y)}{\partial x}\cdot dx + \frac{\partial F(x,y)}{\partial y}\cdot dy = 0 \quad \rightarrow \quad \frac{dy}{dx} = -\frac{\left(\dfrac{\partial F(x,y)}{\partial x}\right)}{\left(\dfrac{\partial F(x,y)}{\partial y}\right)} = \tan\left(\alpha - \frac{\pi}{2}\right) = \frac{-1}{\tan(\alpha)}$$

The last equation gives the normal direction angle at point E, α, and the conformal mapping function, ω(ζ), is used to convert the result to α(ζ). Also note that in terms of arc length, s, along the ξ = constant line,

$$\frac{dx}{ds} = \sin(\alpha)\,, \qquad \frac{dy}{ds} = \cos(\alpha)$$

As an example of finding α, the mapping for a circle of radius R in the z-plane is,

$$z = R\cdot e^{i\cdot\hat{\alpha}} = \omega(\zeta) = R\cdot\zeta = R\cdot\rho\cdot e^{i\cdot\theta}$$

$$F(x,y) = x^2 + y^2 - R^2$$

$$dx + i\cdot dy = R\cdot\rho\cdot i\cdot e^{i\cdot\theta} = i\cdot R\cdot\rho\cdot(\cos(\theta) + i\cdot\sin(\theta))\cdot d\theta$$

$$dx = -R\cdot\rho\cdot\sin(\theta)\cdot d\theta\,, \quad dy = R\cdot\rho\cdot\cos(\theta)\cdot d\theta$$

$$\frac{-1}{\tan(\alpha)} = \frac{dy}{dx} = -\frac{\cos(\theta)}{\sin(\theta)} = \frac{-1}{\tan(\theta)} \quad \rightarrow \quad \tan(\alpha) = \tan(\theta)\,, \quad \alpha = \theta$$

Note that for a circular hole in the z-plane α is independent of ρ.

As a second example, the mapping for an ellipse is, with, major axis in x-direction = $R \cdot (1+m)$, minor axis in y-direction = $R \cdot (1-m)$,

$$z = R \cdot e^{i \cdot \hat{\alpha}} = \omega(\zeta) = R \cdot \left(\zeta + \frac{m}{\zeta} \right) = R \cdot \left(\rho \cdot e^{i \cdot \theta} + \frac{m \cdot e^{-i \cdot \theta}}{\rho} \right)$$

$$F(x, y) = \frac{x^2}{(1+m)^2} + \frac{y^2}{(1-m)^2} - R^2 \quad \rightarrow \quad \frac{dx}{dy} = -\frac{(1+m)^2}{(1-m)^2} \cdot \frac{y}{x} = -\tan(\alpha)$$

$$\rightarrow \quad \tan(\alpha) = \frac{(1+m)^2}{(1-m)^2} \cdot \frac{\mathrm{Im}(z)}{\mathrm{Re}(z)} = \frac{(1+m)^2}{(1-m)^2} \cdot \frac{\rho - \frac{m}{\rho}}{\rho + \frac{m}{\rho}} \cdot \frac{\sin(\theta)}{\cos(\theta)} = \frac{(1+m)^2}{(1-m)^2} \cdot \frac{\rho - \frac{m}{\rho}}{\rho + \frac{m}{\rho}} \cdot \tan(\theta)$$

$$\rightarrow \quad \alpha = \arctan\left[\frac{(1+m)^2}{(1-m)^2} \cdot \frac{\rho - \frac{m}{\rho}}{\rho + \frac{m}{\rho}} \cdot \tan(\theta) \right]$$

For the ellipse the value of α depends on ρ and θ. On the boundary of the hole ($\rho = 1$),

$$\tan(\alpha)\Big|_{\text{boundary}} = \frac{1+m}{1-m} \cdot \tan(\theta) \quad \rightarrow \quad \alpha\Big|_{\text{boundary}} = \arctan\left[\frac{1+m}{1-m} \cdot \tan(\theta) \right]$$

The next step in the method of Muskhelishvili uses the loading on the hole boundary to find the real functions $F_x(s)$, $F_y(s)$ and M, the forces and moment, as,

$$F_x(s) = \int_A^B \overline{X} \cdot ds = \int_A^B \left(\sigma_x \cdot \cos(\alpha) + \tau_{xy} \cdot \sin(\alpha) \right) \cdot ds$$

$$= \int_A^B \left(\frac{\partial^2 \Phi}{\partial y^2} \cdot \frac{dy}{ds} + \frac{\partial^2 \Phi}{\partial x \, \partial y} \cdot \frac{dx}{ds} \right) \cdot ds = \int_A^B \left(\frac{d}{ds} \left(\frac{\partial \Phi}{\partial y} \right) \right) \cdot ds = \left[\frac{\partial \Phi}{\partial y} \right]_A^B$$

$$F_y(s) = \int_A^B \overline{Y} \cdot ds = \int_A^B \left(\sigma_y \cdot \sin(\alpha) + \tau_{xy} \cdot \cos(\alpha) \right) \cdot ds$$

$$= \int_A^B \left(-\frac{\partial^2 \Phi}{\partial x^2} \cdot \frac{dx}{ds} - \frac{\partial^2 \Phi}{\partial x \, \partial y} \cdot \frac{dy}{ds} \right) \cdot ds = \int_A^B - \left(\frac{d}{ds} \left(\frac{\partial \Phi}{\partial y} \right) \right) \cdot ds = -\left[\frac{\partial \Phi}{\partial x} \right]_A^B$$

$$M(s) = \int_A^B x \cdot \overline{Y} \cdot ds - \int_A^B y \cdot \overline{X} \cdot ds = \int_A^B \left[x \cdot \left(\sigma_x \cdot \cos(\alpha) + \tau_{xy} \cdot \sin(\alpha) \right) - y \cdot \left(\sigma_y \cdot \sin(\alpha) + \tau_{xy} \cdot \cos(\alpha) \right) \right] \cdot ds$$

$$= \left[\Phi - x \cdot \frac{\partial \Phi}{\partial x} - y \cdot \frac{\partial \Phi}{\partial y} \right]_A^B$$

where s is length measured along the hole boundary and the integrations are in the z-plane. The details of the M(s) integration are not given here as M(s) in not used explicitly in this report. The integrals are force and moment components on the hole boundary between the two points A and B. For the problems considered here the boundary point A is taken at $\alpha = 0$ and point B is taken as a generic value of increased α. In general, $F_x(s)$ and $F_y(s)$ are real and depend on position along the boundary in the z-plane. The conformal mapping function $\omega(\sigma)$ can be used to express these forces as functions of σ. That is, as $F_x(\sigma)$ and $F_y(\sigma)$ and define,

$$f(\sigma) = i \cdot \left(F_x(\sigma) + i \cdot F_y(\sigma)\right) = -i \cdot \left[\frac{\partial \Phi}{\partial x} + i \cdot \frac{\partial \Phi}{\partial y}\right] = -i \cdot \left[\varphi(z) + z \cdot \overline{\varphi}'(z) + \overline{\psi}'(\overline{z})\right]_A^B$$

This last equation is the form for the loading used in this procedure.

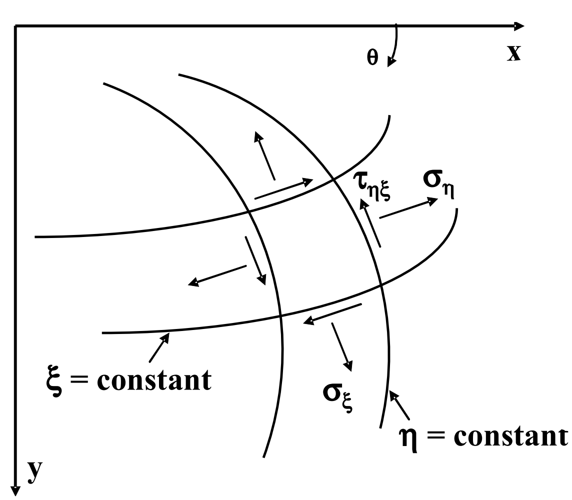
The two harmonic functions that determine the solution are $\varphi(z)$ and $\psi(z)$. These functions are used to determine the stresses and displacements as follows.

$$\sigma_\xi + \sigma_\eta = 4 \cdot \mathrm{Re}\left[\varphi'(z)\right]$$

$$\sigma_\eta - \sigma_\xi + 2 \cdot i \cdot \tau_{\eta\xi} = 2 \cdot e^{2 \cdot i \cdot \alpha} \cdot \left[\overline{z} \cdot \varphi''(z) + \psi'(z)\right]$$

$$2 \cdot G \cdot \left(u_\xi - i \cdot u_\eta\right) = e^{i \cdot \alpha} \cdot \left[\frac{3-\nu}{1+\nu} \cdot \overline{\varphi}(\overline{z}) - \overline{z} \cdot \varphi'(z) - \psi(z)\right]$$

where G is the shear modulus and ν is Poisson's ratio for the material. In this report a prime symbol on a function implies taking the derivative of the function with respect to its argument. The sketch below shows the positive convention for stresses σ_ξ, σ_η and $\tau_{\xi\eta}$. The corresponding displacements are u_ξ and u_η directed in the increasing coordinate direction. All of the above equations are for the plane stress condition. Every elasticity textbook shows the equations for plane strain are obtained by replacing $\frac{3-\nu}{1+\nu}$ in the above equations by $3 - 4 \cdot \nu$.



The expressions for the stresses and displacements given above contain φ(z) and its derivatives so that the mapping function z = ω(ζ) is used to change the variable. For example,

$$\frac{\partial\,\varphi(z)}{\partial z} = \frac{\partial\,\varphi(\omega(\zeta))}{\partial z} = \varphi'(\zeta)\cdot\frac{\partial\zeta}{\partial z} = \varphi'(\zeta)\cdot\frac{1}{\omega'(\zeta)}$$

$$\frac{\partial^2\,\varphi(z)}{\partial z^2} = \frac{\partial}{\partial z}\left(\varphi'(\zeta)\cdot\frac{1}{\omega'(\zeta)}\right) = \frac{\partial}{\partial\zeta}\left(\varphi'(\zeta)\cdot\frac{1}{\omega'(\zeta)}\right)\cdot\frac{1}{\omega'(\zeta)} = \varphi''(\zeta)\cdot\frac{1}{(\omega'(\zeta))^2} + \varphi'(\zeta)\cdot\frac{-\omega''(\zeta)}{(\omega'(\zeta))^3}$$

also,

$$\frac{\partial\,\psi(z)}{\partial z} = \frac{\partial\,\psi(\omega(\zeta))}{\partial z} = \psi'(\zeta)\cdot\frac{\partial\zeta}{\partial z} = \psi'(\zeta)\cdot\frac{1}{\omega'(\zeta)}$$

The values of F_x and F_y can be expressed in terms of φ(z) and ψ(z) as shown above,

$$F_x + i\cdot F_y = -i\cdot\left[\varphi(z) + z\cdot\overline{\varphi}(\overline{z}) + \overline{\psi}(\overline{z})\right]_A^B = -i\cdot f(\sigma)$$

so that on the hole boundary in the ζ-plane,

$$\varphi(\sigma) + \frac{\omega(\sigma)}{\overline{\omega}'(\overline{\sigma})}\cdot\overline{\varphi}'(\overline{\sigma}) + \overline{\psi}(\overline{\sigma}) = f(\sigma)$$

When both harmonic functions are known the equations given above can be used to find the stresses and displacements. It is clear that this application requires only that appropriate F_x, F_y and ω(ζ) be specified; the rest is mathematical manipulation.

In order to determine φ(ζ) and ψ(ζ) the boundary conditions must be imposed. The last equation above is the first part of the boundary conditions. The other part of the boundary conditions is the conjugate of the last equation which is,

$$\overline{\varphi}(\overline{\sigma}) + \frac{\overline{\omega}(\overline{\sigma})}{\omega'(\sigma)}\cdot\varphi'(\sigma) + \psi(\sigma) = \bar{f}(\overline{\sigma})$$

Let ζ be a point outside the unit circle in the ζ-plane. The procedure for finding φ(ζ) and ψ(ζ) is to divide each of the last two equations by σ − ζ and then integrate around the unit circle. The two resulting equations are,

$$\oint_\gamma \frac{\varphi(\sigma)\cdot d\sigma}{\sigma-\zeta} + \oint_\gamma \frac{\omega(\sigma)}{\overline{\omega}'(\overline{\sigma})}\cdot\frac{\overline{\varphi}'(\overline{\sigma})\cdot d\sigma}{\sigma-\zeta} + \oint_\gamma \frac{\overline{\psi}(\overline{\sigma})\cdot d\sigma}{\sigma-\zeta} = \oint_\gamma \frac{f(\sigma)\cdot d\sigma}{\sigma-\zeta}$$

A

$$\oint_\gamma \frac{\overline{\varphi}(\overline{\sigma})\cdot d\sigma}{\sigma-\zeta} + \oint_\gamma \frac{\overline{\omega}(\overline{\sigma})}{\omega'(\sigma)}\cdot\frac{\varphi'(\sigma)\cdot d\sigma}{\sigma-\zeta} + \oint_\gamma \frac{\psi(\sigma)\cdot d\sigma}{\sigma-\zeta} = \oint_\gamma \frac{\overline{f}(\overline{\sigma})\cdot d\sigma}{\sigma-\zeta}$$

B

The Cauchy theorems are used to evaluate the terms in Equations A and B. These evaluations are described in Appendix A of this report. The resulting equations are,

$$\varphi(\zeta) = -\frac{1}{2\cdot\pi\cdot i}\cdot\int_\sigma \frac{f(\sigma)\cdot d\sigma}{\sigma-\zeta}$$

$$\psi(\zeta) = -\frac{1}{2\cdot\pi\cdot i}\cdot\int_\gamma \frac{\overline{f}(\overline{\sigma})\cdot d\sigma}{\sigma-\zeta} + \frac{1}{2\cdot\pi\cdot i}\cdot\int_\sigma \frac{\overline{\omega}(\overline{\sigma})}{\omega'(\sigma)}\cdot\frac{\cdot\varphi'(\sigma)}{\sigma-\zeta}\cdot d\sigma$$

When f(σ) and ω(σ) are specified, the first equation determines φ(ζ) and then the second equation determines ψ(ζ). The stresses and displacements are found using equations given above.

UNLOADED CIRCULAR HOLE BOUNDARY IN AN INFINITE PLATE WITH TENSION IN THE X-DIRECTION AT INFINITY

Consider first the plate without the hole. The only non-vanishing stress component everywhere is the normal stress in the x-direction, $\sigma_x O$. The center for the hole is at the origin of the x, y coordinates and its radius is R. When the hole is introduced the stresses will be unaltered in the remainder of the plate if the hole boundary has the following stresses applied, in r- $\hat{\alpha}$ polar coordinates (note that $\hat{\alpha}$ is measured positive clockwise from the positive x-direction),

$$\sigma_\xi = \tfrac{1}{2}\cdot\sigma_x O\cdot(1+\cos(2\cdot\hat{\alpha}))$$

$$\tau_{\eta\xi} = -\tfrac{1}{2}\cdot\sigma_x O\cdot\sin(2\cdot\hat{\alpha})$$

and the hoop stress, σ_η, at the hole boundary is,

$$\sigma_\eta = \tfrac{1}{2}\cdot\sigma_x O\cdot(1-\cos(2\cdot\hat{\alpha}))$$

If the negatives of σ_ξ and $\tau_{\eta\xi}$ are superposed on the hole boundary, the boundary loading vanishes and the desired stress distribution is achieved.

The Muskhelishvili procedure is applied in this section to remove the above hole boundary loads while ensuring the stress at infinity is maintained at the x-direction tension. In other words, except for σ_x, the stresses resulting from this procedure after superposing all vanish at infinity.

The conformal mapping that maps a circle in the z-plane (complex plane) into another circle in the ζ-plane is,

$$z = x + i\cdot y = r\cdot e^{1\cdot\hat{\alpha}} \qquad \zeta = \xi + i\cdot\eta = \rho\cdot e^{i\cdot\theta}$$

$$z = \omega(\zeta) = R\cdot\zeta$$

The objective here is to map z for the hole boundary of radius R onto the unit circle defined by $\sigma = e^{i\cdot\theta}$ so that $\rho = 1$ and from Page 4, $\alpha = \theta$. Therefore the boundary mapping is,

$$z\big|_{\text{hole boundary}} = \omega(\sigma) = R\cdot\sigma = R\cdot e^{i\cdot\theta}$$

In the following procedure the loads on the hole boundary are defined in terms of $\overline{X}$ and $\overline{Y}$, the forces per unit boundary length as described earlier. Let s be a curvilinear coordinate along the hole boundary (s is a complex number along the hole boundary in the z-plane), then, Integral of force vector in x-direction between points A and B $\equiv F_x = \int_A^B \overline{X} \cdot ds$

Integral of force vector in y-direction between points A and B $\equiv F_y = \int_A^B \overline{Y} \cdot ds$

For this problem the hole is a closed boundary and there is symmetry about the x-axis. F_x and F_y are determined choosing $\alpha = 0$ for point A and a generic value of α for point B so that,

$$
\begin{aligned}
F_x &= \int_{s=0}^{s(\alpha)} \left(\sigma_x \cdot \cos(\alpha) + \tau_{xy} \cdot \sin(\alpha)\right) \cdot ds \\
&= \tfrac{1}{2} \cdot \sigma_x O \cdot R \cdot \int_{\alpha=0}^{\alpha} \left[(1+\cos(2\cdot\alpha)) \cdot \cos(\alpha) - \sin(2\cdot\alpha) \cdot \sin(\alpha)\right] \cdot d\alpha \\
&= \sigma_x O \cdot R \cdot \sin(\alpha)
\end{aligned}
$$

$$
\begin{aligned}
F_y &= \int_{s=0}^{s(\alpha)} \left(\sigma_x \cdot \sin(\alpha) - \tau_{xy} \cdot \cos(\alpha)\right) \cdot ds \\
&= \tfrac{1}{2} \cdot \sigma_x O \cdot R \cdot \int_{\theta=0}^{\theta} \left[(1+\cos(2\cdot\alpha)) \cdot \sin(\alpha) + \sin(2\cdot\alpha) \cdot \cos(\alpha)\right] \cdot d\alpha \\
&= 0
\end{aligned}
$$

and then,

$$
F_x + i \cdot F_y = \sigma_x O \cdot R \cdot \sin(\alpha) = \sigma_x O \cdot R \cdot \frac{e^{i\cdot\alpha} - e^{-i\cdot\alpha}}{2\cdot i} = \frac{1}{2\cdot i} \cdot \sigma_x O \cdot R \cdot \left(\sigma - \frac{1}{\sigma}\right)
$$

The negative of this boundary loading is applied and the Muskhelishvili procedure is used to find the solution to this auxiliary problem.

The first step is to define $f(\sigma)$ corresponding to the boundary loading as,

$$
f(\sigma) = -i \cdot \left(F_x + i \cdot F_y\right) = -\tfrac{1}{2} \cdot \sigma_x O \cdot R \cdot \left(\sigma - \frac{1}{\sigma}\right)
$$

This boundary loading is now used in the following integral and Cauchy's residue theorem is applied to obtain the first of the two potential functions, $\varphi(\zeta)$, required for the solution,

$$\varphi(\zeta) = -\frac{1}{2\cdot\pi\cdot i}\cdot\int_{\gamma}\frac{f(\sigma)\cdot d\sigma}{\sigma-\zeta} = \tfrac{1}{2}\cdot\sigma_x O\cdot R\cdot\frac{1}{\zeta}$$

since f(σ) has a simple pole (m_k =1) at σ = 0, the residual is,,

$$\mathrm{Res}(0) = 1\cdot\left[(\sigma-0)\cdot\frac{-\tfrac{1}{2}\cdot\sigma_x O\cdot R\cdot\left(\sigma-\dfrac{1}{\sigma}\right)}{\sigma-\zeta}\right]_{\sigma=0} = \left(\frac{-\tfrac{1}{2}\cdot\sigma_x O\cdot R\cdot(\sigma^2-1)}{\sigma-\zeta}\right)_{\sigma=0} = -\frac{\tfrac{1}{2}\cdot\sigma_x O\cdot R}{\zeta}$$

The second potential function, ψ(ζ), is found using,

$$\bar{f}(\bar{\sigma}) = f\left(\frac{1}{\sigma}\right) = -\tfrac{1}{2}\cdot\sigma_x O\cdot R\cdot\left(\frac{1}{\sigma}-\sigma\right)$$

$$\varphi'(\zeta) = -\tfrac{1}{2}\cdot\sigma_x O\cdot R\cdot\frac{1}{\zeta^2}$$

in the definition of ψ(ζ) which is,

$$\psi(\zeta) = -\frac{1}{2\cdot\pi\cdot i}\cdot\int_{\gamma}\frac{\bar{f}(\bar{\sigma})\cdot d\sigma}{\sigma-\zeta} + \frac{1}{2\cdot\pi\cdot i}\cdot\int_{\sigma}\frac{\overline{\omega}(\bar{\sigma})}{\omega'(\sigma)}\cdot\frac{\cdot\varphi'(\sigma)}{\sigma-\zeta}\cdot d\sigma$$

and

$$\frac{\overline{\omega}(\bar{\sigma})}{\omega'(\sigma)} = \bar{\sigma} = \frac{1}{\sigma}$$

so that, after using Cauchy's residue theorem,

$$\psi(\zeta) = \tfrac{1}{2}\cdot\sigma_x O\cdot R\cdot\left(\frac{1}{\zeta}+\frac{1}{\zeta^3}\right)$$

The first term in the above equation is from the first integral and the integration is analogous to the integration for φ(σ) while the second term is from the second integral that has a third order pole (m_k = 3) at σ = 0 so the residue is obtained as follows,

$$\mathrm{Res}(0) = \frac{1}{2!}\cdot\left[\frac{d^2}{d\sigma^2}\left\{(\sigma-0)^3\cdot\left(\frac{-\tfrac{1}{2}\cdot\sigma_x O\cdot R}{\sigma^3\cdot(\sigma-\zeta)}\right)\right\}\right]_{\sigma=0} = -\tfrac{1}{4}\cdot\sigma_x O\cdot R\cdot\left[\frac{d^2}{d\sigma^2}\left\{\frac{1}{\sigma-\zeta}\right\}\right]_{\sigma=0}$$

$$= -\tfrac{1}{2}\cdot\sigma_x O\cdot R\cdot\left[\frac{1}{(\sigma-\zeta)^3}\right]_{\sigma=0} = \frac{\tfrac{1}{2}\cdot\sigma_x O\cdot R}{\zeta^3}$$

This determination of the potential functions, φ(ζ) and ψ(ζ), completes the formal part of the Muskhelishvili procedure.

The final step is to determine the stresses and displacements with the following equations,

$$\sigma_\xi + \sigma_\eta = 4 \cdot \mathrm{Re}[\varphi'(z)]$$

$$\sigma_\eta - \sigma_\xi + 2 \cdot i \cdot \tau_{\eta\xi} = 2 \cdot e^{2 \cdot i \cdot \alpha} \cdot [\bar{z} \cdot \varphi''(z) + \psi'(z)]$$

$$2 \cdot G \cdot (u_\xi - i \cdot u_\eta) = e^{i \cdot \alpha} \cdot \left[\frac{3-\nu}{1+\nu} \cdot \overline{\varphi}(\bar{z}) - \bar{z} \cdot \varphi'(z) - \psi(z)\right]$$

where,

α = clockwise angle from the x-axis to the outward normal to ξ = constant line in z-plane
= θ in this problem as was shown earlier

A prime indicates differentiation with respect to the indicated independent variable, that is,

$$\varphi'(z) = \varphi'(\zeta) \cdot \frac{\partial \zeta}{\partial z} = \frac{\varphi'(\zeta)}{R}$$

$$\varphi''(z) = \varphi''(\zeta) \cdot \left(\frac{\partial \zeta}{\partial z}\right)^2 + \varphi'(\zeta) \cdot \frac{\partial^2 \zeta}{\partial z^2} = \frac{\varphi''(\zeta)}{R^2}$$

$$\psi'(z) = \psi'(\zeta) \cdot \frac{\partial \zeta}{\partial z} = \frac{\psi'(\zeta)}{R}$$

Therefore, the stresses and displacements may be written as,

$$\sigma_\xi + \sigma_\eta = \frac{4}{R} \cdot \mathrm{Re}[\varphi'(\zeta)]$$

$$\sigma_\eta - \sigma_\xi + 2 \cdot i \cdot \tau_{\eta\xi} = \frac{2 \cdot e^{2 \cdot i \cdot \theta}}{R} \cdot [\bar{\zeta} \cdot \varphi''(\zeta) + \psi'(\zeta)]$$

$$2 \cdot G \cdot (u_\xi - i \cdot u_\eta) = e^{i \cdot \theta} \cdot \left[\frac{3-\nu}{1+\nu} \cdot \overline{\varphi}(\bar{\zeta}) - \bar{\zeta} \cdot \varphi'(\zeta) - \psi(\zeta)\right]$$

Substituting φ(ζ) and ψ(ζ) into the three above equations, eliminating ζ using $z = R \cdot \zeta$, and setting $z = r \cdot e^{i \cdot \alpha}$ yields,

$$\sigma_\xi + \sigma_\eta = -2\cdot\sigma_x O\cdot \mathrm{Re}\left[\frac{1}{\zeta^2}\right] = -2\cdot\sigma_x O\cdot \mathrm{Re}\left[\frac{R^2}{r^2}\cdot e^{-2\cdot i\cdot\alpha}\right] = -2\cdot\sigma_x O\cdot\frac{R^2}{r^2}\cdot\cos(2\cdot\alpha)$$

$$\sigma_\eta - \sigma_\xi + 2\cdot i\cdot\tau_{\eta\xi} = \frac{2\cdot e^{2\cdot i\cdot\theta}}{R}\cdot\left[\bar{\zeta}\cdot\varphi''(\zeta) + \psi'(\zeta)\right] = \sigma_x O\cdot e^{2\cdot i\cdot\alpha}\cdot\left[\frac{2\cdot\bar{\zeta}}{\zeta^3} + \frac{1}{\zeta^2} - \frac{3}{\zeta^4}\right]$$

$$= \sigma_x O\cdot\left[\frac{R^2}{r^2} + \left(2\cdot\frac{R^2}{r^2} - 3\cdot\frac{R^4}{r^4}\right)\cdot e^{-2\cdot i\cdot\alpha}\right]$$

$$= \sigma_x O\cdot\left\{\left[\frac{R^2}{r^2} + \left(2\cdot\frac{R^2}{r^2} - 3\cdot\frac{R^4}{r^4}\right)\cdot\cos(2\cdot\alpha)\right] - i\cdot\left(2\cdot\frac{R^2}{r^2} - 3\cdot\frac{R^4}{r^4}\right)\cdot\sin(2\cdot\alpha)\right\}$$

$$2\cdot G\cdot\left(u_\xi - i\cdot u_\eta\right) = e^{i\cdot\alpha}\cdot\left[\frac{3-\nu}{1+\nu}\cdot\bar{\varphi}(\bar{\zeta}) - \bar{\zeta}\cdot\varphi'(\zeta) - \psi(\zeta)\right]$$

$$= \frac{\sigma_x O\cdot R}{2}\cdot\left[\frac{3-\nu}{1+\nu}\cdot\frac{R}{r}\cdot e^{2\cdot i\cdot\alpha} + \frac{R}{r}\cdot e^{-2\cdot i\cdot\alpha} + \frac{R}{r} - \frac{R^2}{r^2}\cdot e^{-i\cdot\theta}\right]$$

$$= \frac{\sigma_x O\cdot R}{2}\cdot\left\{\begin{aligned}&\left[\frac{R}{r}\cdot\left(1+\frac{4}{1+\nu}\cdot\cos(2\cdot\alpha)\right) - \frac{R^2}{r^2}\cdot\cos(\alpha)\right]\\ &+ i\cdot\left[\frac{R}{r}\cdot\left(-1+2\cdot\frac{1-\nu}{1+\nu}\cdot\sin(2\cdot\alpha)\right) - \frac{R^2}{r^2}\cdot\sin(\alpha)\right]\end{aligned}\right\}$$

The three equations above are used to find σ_ξ, σ_η, $\tau_{\eta\xi}$, u_η and u_ξ which are to be superposed on the conditions prevailing without the hole given earlier with the following results,

$$\sigma_\xi = \tfrac{1}{2}\cdot\sigma_x O\cdot\left[\frac{R^2}{r^2}\cdot(-1 - 4\cdot\cos(2\cdot\alpha) + 3\cdot\frac{R^4}{r^4}\cdot\cos(2\cdot\alpha)\right]$$

$$\sigma_\eta = \tfrac{1}{2}\cdot\sigma_x O\cdot\left(\frac{R^2}{r^2} - 3\cdot\frac{R^4}{r^4}\cdot\cos(2\cdot\alpha)\right)$$

$$\tau_{\xi\eta} = -\tfrac{1}{2}\cdot\sigma_x O\cdot\left(2\cdot\frac{R^2}{r^2} - 3\cdot\frac{R^4}{r^4}\right)\cdot\sin(2\cdot\alpha)$$

$$2\cdot G\cdot u_\xi = \frac{\sigma_x O\cdot R}{2}\cdot\left[\frac{R}{r}\cdot\left(1+\frac{4}{1+\nu}\cdot\cos(2\cdot\alpha)\right) - \frac{R^2}{r^2}\cdot\cos(\alpha)\right]$$

$$2 \cdot G \cdot u_{\eta} = -\frac{\sigma_x O \cdot R}{2} \cdot \left[\frac{R}{r} \cdot \left(1 - 2 \cdot \frac{1-\nu}{1+\nu} \cdot \sin(2 \cdot \alpha) \right) + \frac{R^2}{r^2} \cdot \sin(\alpha) \right]$$

When the above stresses are added to the stresses in the first three equations of this section, the desired stresses are found and given below,

$$\sigma_r = \tfrac{1}{2} \cdot \sigma_x O \cdot \left[1 - \frac{R^2}{r^2} + \cos(2 \cdot \alpha) \cdot \left(1 - 4 \cdot \frac{R^2}{r^2} + 3 \cdot \frac{R^4}{r^4} \right) \right]$$

$$\sigma_\theta = \tfrac{1}{2} \cdot \sigma_x O \cdot \left[1 + \frac{R^2}{r^2} - \cos(2 \cdot \alpha) \cdot \left(1 + 3 \cdot \frac{R^4}{r^4} \right) \right]$$

$$\tau_{r\theta} = -\tfrac{1}{2} \cdot \sigma_x O \cdot \left(1 + 2 \cdot \frac{R^2}{r^2} - 3 \cdot \frac{R^4}{r^4} \right) \cdot \sin(2 \cdot \alpha)$$

The displacement field given above contains a rigid body motion that does not contribute to the strains and is omitted below. In addition, the displacements owing to the uniform strain field caused by the uniform stress field, $\sigma_x O$, are not included below. With these provisions the displacements are,

$$2 \cdot G \cdot u_r = \frac{\sigma_x O \cdot R}{2} \cdot \frac{R}{r} \cdot \left(1 + \frac{4}{1+\nu} \cdot \cos(2 \cdot \alpha) \right)$$

$$2 \cdot G \cdot u_\theta = -\frac{\sigma_x O \cdot R}{2} \cdot \frac{R}{r} \cdot \left(-1 + 2 \cdot \frac{1-\nu}{1+\nu} \cdot \sin(2 \cdot \alpha) \right)$$

UNLOADED ELLIPTICAL HOLE BOUNDARY IN AN INFINITE PLATE WITH TENSION IN THE X-DIRECTION AT INFINITY

A mapping that takes an ellipse in the z-plane to a unit circle in the ζ-plane is,

$$z = \omega(\zeta) = R \cdot \left(\zeta + \frac{m}{\zeta}\right) \equiv r \cdot e^{i \cdot \hat{\alpha}} \qquad \hat{\alpha} = \arctan\left(\frac{\mathrm{Im}(z)}{\mathrm{Re}(z)}\right) = \arctan\left(\frac{y}{x}\right)$$

where the ellipse has a semi-axis of length $R \cdot (1+m)$ in the x-direction and its other semi-axis has a length of $R \cdot (1-m)$ in the y-direction. The radius r and the angle $\hat{\alpha}$ are the usual polar coordinates in the z-plane. Since the hole boundary in the ζ-plane is $e^{i \cdot \theta} \equiv \sigma$, the hole boundary in the z-plane is given by,

$$z\big|_{\text{boundary}} = \omega(\sigma) = R \cdot \left(\sigma + \frac{m}{\sigma}\right)$$

A few useful results for this problem are,

$$\omega'(\sigma) = R \cdot \left(1 - \frac{m}{\sigma^2}\right)$$

$$\omega''(\sigma) = R \cdot \frac{2 \cdot m}{\sigma^3}$$

$$\overline{\omega}(\overline{\sigma}) = R \cdot \left(\frac{1}{\sigma} + m \cdot \sigma\right)$$

$$\overline{\omega}'(\overline{\sigma}) = R \cdot \left(1 - m \cdot \sigma^2\right)$$

$$\frac{\omega(\sigma)}{\overline{\omega}'(\overline{\sigma})} = \frac{1}{\sigma} \cdot \frac{\sigma^2 + m}{1 - m \cdot \sigma^2}$$

$$\frac{\overline{\omega}(\overline{\sigma})}{\omega'(\sigma)} = \frac{\sigma \cdot \left(1 + m \cdot \sigma^2\right)}{\sigma^2 - m}$$

$$\frac{\omega''(\sigma)}{\left(\omega'(\sigma)\right)^3} = \frac{2 \cdot m}{R^2 \cdot \left(\sigma - \frac{m}{\sigma}\right)^3}$$

$$\alpha = \arctan\left[\frac{(1+m)^2}{(1-m)^2}\cdot\frac{\rho-\frac{m}{\rho}}{\rho+\frac{m}{\rho}}\cdot\tan(\theta)\right] \qquad \text{(from Page 5)}$$

The stresses on the boundary before the elliptical hole is introduced are,

$$\sigma_\eta = \tfrac{1}{2}\cdot\sigma_x O\cdot(1+\cos(2\cdot\alpha))$$

$$\tau_{\eta\xi} = -\tfrac{1}{2}\cdot\sigma_x O\cdot\sin(2\cdot\alpha)$$

and the hoop stress at the hole boundary is,

$$\sigma_\xi = \tfrac{1}{2}\cdot\sigma_x O\cdot(1-\cos(2\cdot\alpha))$$

Rather than follow the integrating procedure employed for the previous problem to obtain $F_x + i\cdot F_y$, advantage is taken of an obvious "shortcut" given by,

$$F_x + i\cdot F_y = \sigma_x O\cdot\frac{1}{2\cdot i}\cdot\left(z_{boundary} - \overline{z}_{boundary}\right) = \frac{\sigma_x O\cdot R}{2\cdot i}\cdot\left(\sigma+\frac{m}{\sigma}-\frac{1}{\sigma}-m\cdot\sigma\right)$$

so that,

$$f(\sigma) = -i\cdot\left(F_x + i\cdot F_y\right) = -\tfrac{1}{2}\cdot\sigma_x O\cdot R\cdot(1-m)\cdot\left(\sigma-\frac{1}{\sigma}\right)$$

and,

$$\bar{f}(\bar{\sigma}) = -\tfrac{1}{2}\cdot\sigma_x O\cdot R\cdot(1-m)\cdot\left(\frac{1}{\sigma}-\sigma\right)$$

The equation giving $\varphi(\zeta)$ may be determined now by using the Cauchy residue theorem as follows,

$$\varphi(\zeta) = -\frac{1}{2\cdot\pi\cdot i}\cdot\int_\sigma \frac{-\tfrac{1}{2}\cdot\sigma_x O\cdot R\cdot(1-m)\cdot\left(\sigma-\frac{1}{\sigma}\right)\cdot d\sigma}{\sigma-\zeta} = \tfrac{1}{2}\cdot\sigma_x O\cdot R\cdot(1-m)\cdot\frac{1}{\zeta}$$

The above integral has a simple pole ($m_k = 1$) at $\sigma = 0$ giving,

$$\mathrm{Res}(0) = 1\cdot\left[(\sigma-0)\cdot\left(\frac{-\frac{1}{2}\cdot\sigma_x O\cdot R\cdot(1-m)\cdot\left(\sigma-\frac{1}{\sigma}\right)}{\sigma-\zeta}\right)\right]_{\sigma=0} = \left[\frac{-\frac{1}{2}\cdot\sigma_x O\cdot R\cdot(1-m)\cdot\left(\sigma^2-1\right)}{\sigma-\zeta}\right]_{\sigma=0}$$

$$= -\tfrac{1}{2}\cdot\sigma_x O\cdot R\cdot(1-m)\cdot\frac{1}{\zeta}$$

then,

$$\varphi'(\zeta) = -\tfrac{1}{2}\cdot\sigma_x O\cdot R\cdot(1-m)\cdot\frac{1}{\zeta^2}$$

and,

$$\varphi''(\zeta) = \sigma_x O\cdot R\cdot(1-m)\cdot\frac{1}{\zeta^3}$$

The equation giving $\psi(\zeta)$ is determined as follows.

$$\psi(\zeta) = -\frac{1}{2\cdot\pi\cdot i}\cdot\int_\gamma\frac{\bar{f}(\bar{\sigma})\cdot d\sigma}{\sigma-\zeta} + \frac{1}{2\cdot\pi\cdot i}\cdot\int_\sigma\frac{\overline{\omega}(\bar{\sigma})}{\omega'(\sigma)}\cdot\frac{\cdot\varphi'(\sigma)}{\sigma-\zeta}\cdot d\sigma$$

$$= -\tfrac{1}{2}\cdot\sigma_x O\cdot R\cdot(1-m)\cdot\frac{1}{\zeta} + \tfrac{1}{2}\cdot\sigma_x O\cdot R\cdot(1-m)\cdot\frac{\left(1+m\cdot\zeta^2\right)}{\zeta\cdot\left(\zeta^2-m\right)}$$

$$= \tfrac{1}{2}\cdot\sigma_x O\cdot R\cdot(1-m)\cdot\left(\frac{\left(1+m\cdot\zeta^2\right)}{\zeta\cdot\left(\zeta^2-m\right)} - \frac{1}{\zeta}\right)$$

so that,

$$\psi'(\zeta) = \tfrac{1}{2}\cdot\sigma_x O\cdot R\cdot(1-m)\cdot\left[\frac{2\cdot m\cdot\zeta^2\cdot\left(\zeta^2-m\right)-\left(1+m\cdot\zeta^2\right)\cdot\left(3\cdot\zeta^2-m\right)+\left(\zeta^2-m\right)^2}{\zeta^2\cdot\left(\zeta^2-m\right)^2}\right]$$

Noting that for any function f(ζ),

$$\frac{\partial f(\zeta)}{\partial z} = \frac{\partial f(\zeta)}{\partial\zeta}\cdot\frac{\partial\zeta}{\partial z} \qquad \text{and} \qquad \frac{\partial^2 f(\zeta)}{\partial z^2} = \frac{\partial^2 f(\zeta)}{\partial\zeta^2}\cdot\left(\frac{\partial\zeta}{\partial z}\right)^2 + \frac{\partial f(\zeta)}{\partial\zeta}\cdot\frac{\partial^2\zeta}{\partial z^2}\cdot\frac{\partial\zeta}{\partial z}$$

The following results are useful,

$$\frac{\partial z}{\partial\zeta} = \omega'(\zeta) \qquad \rightarrow \qquad \frac{\partial\zeta}{\partial z} = \frac{1}{\omega'(\zeta)}$$

$$\frac{\partial^2 \zeta}{\partial z^2} = -\frac{\omega''(\zeta)}{(\omega'(\zeta))^2}\cdot\frac{\partial \zeta}{\partial z} = -\frac{\omega''(\zeta)}{(\omega'(\zeta))^3}$$

so that for the current function $\omega(\zeta)$,

$$\frac{\partial \zeta}{\partial z} = \frac{1}{R\cdot\left(1-\frac{m}{\zeta^2}\right)} \qquad \text{and} \qquad \frac{\partial \zeta^2}{\partial z^2} = -\frac{2\cdot m}{R^2\cdot\left(\zeta-\frac{m}{\zeta}\right)^3}$$

Now,

$$\varphi(\zeta) = \tfrac{1}{2}\cdot\sigma_x O\cdot R\cdot(1-m)\cdot\frac{1}{\zeta}$$

$$\varphi'(z) = -\tfrac{1}{2}\cdot\sigma_x O\cdot R\cdot(1-m)\cdot\frac{1}{\zeta^2}\cdot\frac{1}{R\cdot\left(1-\frac{m}{\zeta^2}\right)}$$

$$\varphi''(z) = \sigma_x O\cdot R\cdot(1-m)\cdot\frac{1}{\zeta^3}\cdot\frac{1}{R^2\cdot\left(1-\frac{m}{\zeta^2}\right)^2} - \tfrac{1}{2}\cdot\sigma_x O\cdot R\cdot(1-m)\cdot\frac{1}{\zeta^2}\cdot\frac{2\cdot m}{R^2\cdot\left(\zeta-\frac{m}{\zeta}\right)^3}\cdot\frac{1}{R\cdot\left(1-\frac{m}{\zeta^2}\right)}$$

$$\psi(\zeta) = \tfrac{1}{2}\cdot\sigma_x O\cdot R\cdot(1-m)\cdot\left(\frac{\left(1+m\cdot\zeta^2\right)}{\zeta\cdot\left(\zeta^2-m\right)}-\frac{1}{\zeta}\right)$$

$$\psi'(z) = \tfrac{1}{2}\cdot\sigma_x O\cdot R\cdot(1-m)\cdot\left[\frac{2\cdot m\cdot\zeta^2\cdot\left(\zeta^2-m\right)-\left(1+m\cdot\zeta^2\right)\cdot\left(3\cdot\zeta^2-m\right)+\left(\zeta^2-m\right)^2}{\zeta^2\cdot\left(\zeta^2-m\right)^2}\right]\cdot\frac{1}{R\cdot\left(1-\frac{m}{\zeta^2}\right)}$$

The stresses and displacements may be found using the above results in,

$$\sigma_\xi + \sigma_\eta = 4\cdot \mathrm{Re}[\varphi'(z)]$$

$$\sigma_\eta - \sigma_\xi + 2\cdot i\cdot\tau_{\eta\xi} = 2\cdot e^{2\cdot i\cdot\alpha}\cdot\left[\bar{z}\cdot\varphi''(z)+\psi'(z)\right]$$

$$2\cdot G\cdot\left(u_\xi + i\cdot u_\eta\right) = e^{i\cdot\alpha}\cdot\left[\frac{3-\nu}{1+\nu}\cdot\overline{\varphi}(\bar{z}) - \bar{z}\cdot\varphi'(z) - \psi(z)\right]$$

In the preceding problem having a circular hole the results were reduced to real expressions for the individual stress and displacement components. For this problem the same procedure would involve considerable algebraic manipulation so a different scheme is recommended and employed here. Most computer compilers are written to accommodate complex numbers. The last three equations are expressed in terms of z,

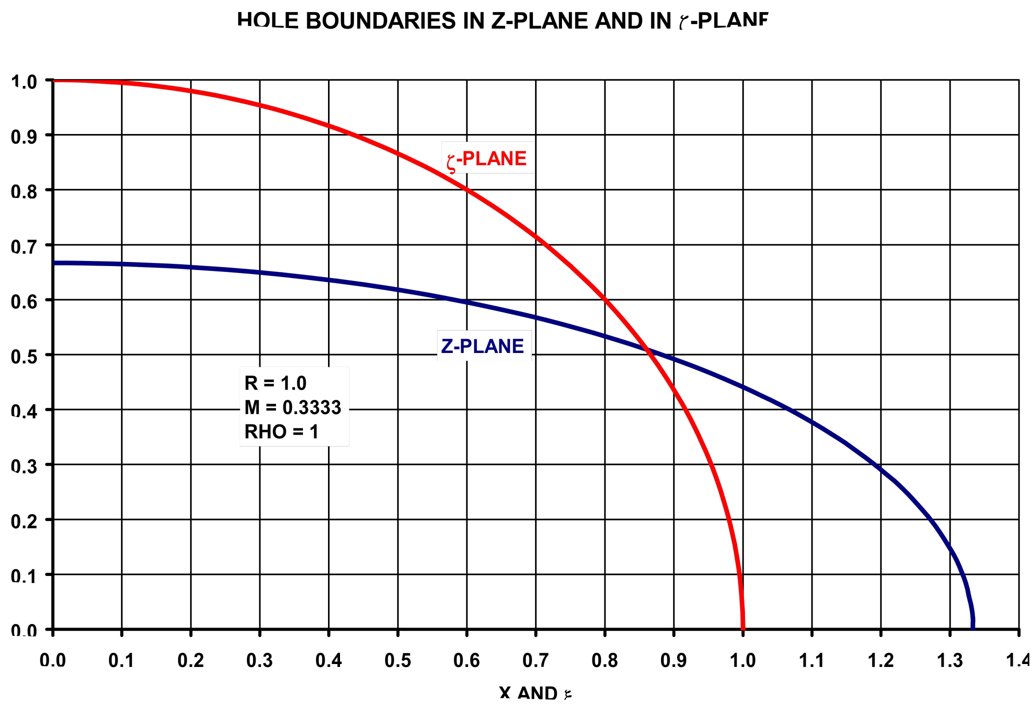
$\varphi(z)$, $\varphi'(z)$, $\varphi''(z)$, $\psi(z)$ and $\psi'(z)$. The functions $\varphi(z)$ and $\psi(z)$ are related to $\varphi(\zeta)$ and $\psi(\zeta)$ through the conformal mapping $\omega(\varphi)$ so all of the functions appearing in the equations for the stress and displacements are known in terms of equations containing complex numbers. When these stresses are found, the stresses for the uniform x-direction normal stress given on page 16 must be added to determine the total stresses and thus the desired solution. The displacements reported here are for the imposed hole boundary stresses and do not include displacements associated with the uniform x-direction stress.

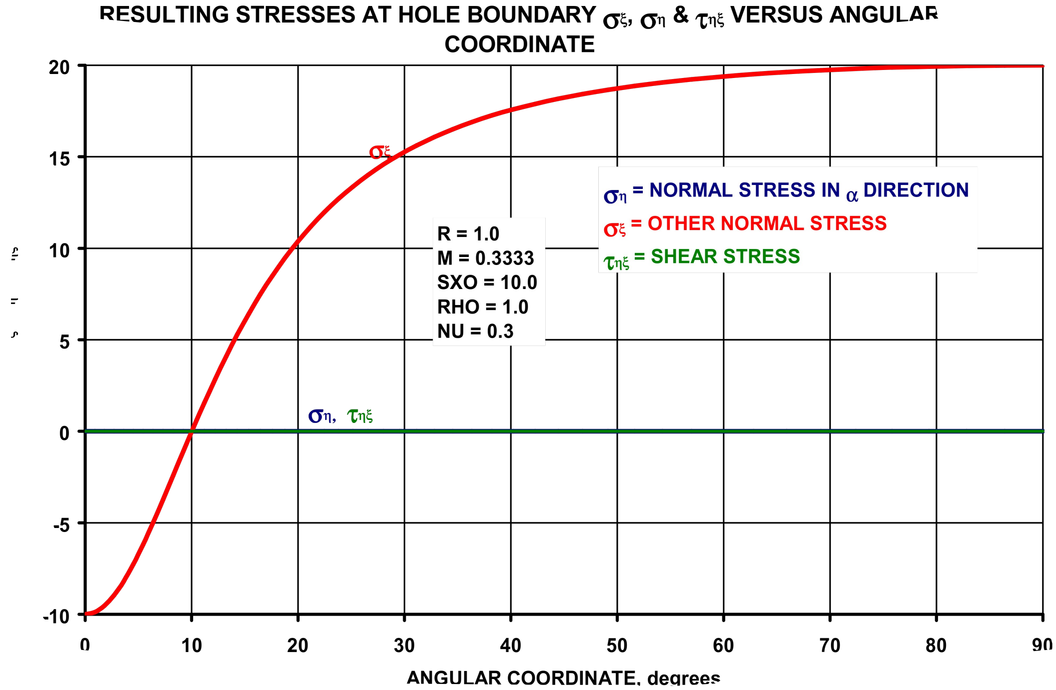
The appendix to this report contains a source listing for a FORTRAN program, MUSK2, which determines the stress and displacement components as described above when the following input data are read from a file.

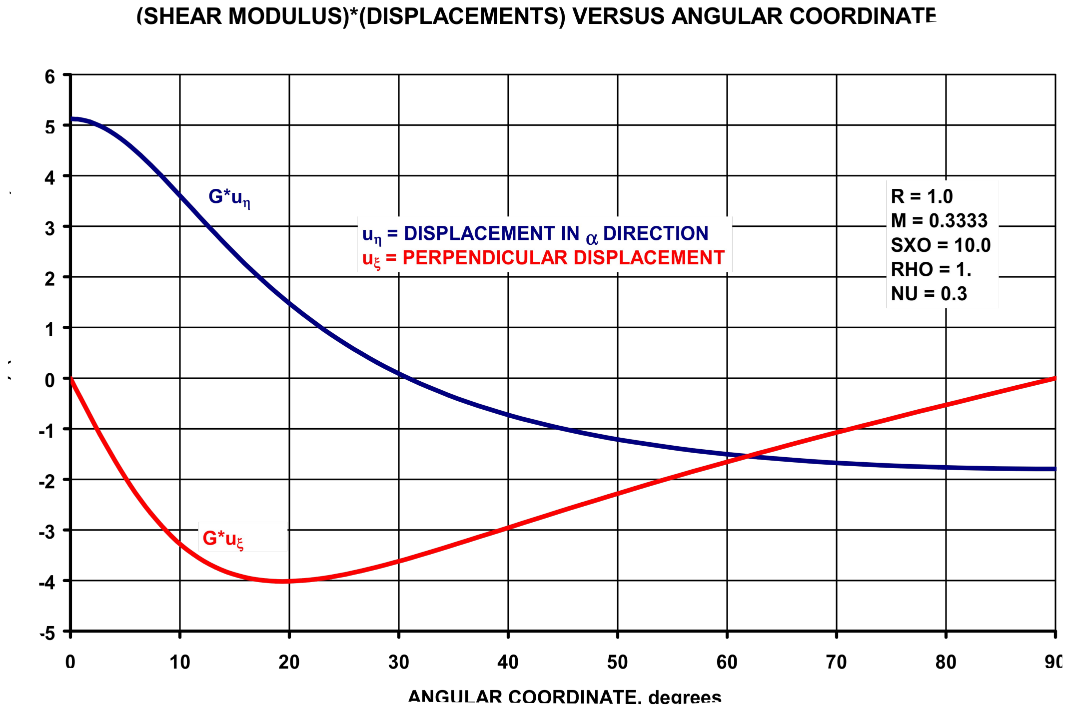
R = mean semi-axis length in z-plane
M = m = ellipse parameter ($-1 < M < 1$), M = 0 for a circular hole
SXO = $\sigma_x O$ = uniform, x-direction normal stress at infinity
RHO = ρ = radius in ζ- plane , (ξ, η plane), (RHO $\geq$ 1), RHO = 1 for hole boundary
NU = ν = Poisson's ratio

The following three figures are from data generated by the computer program for the case where.

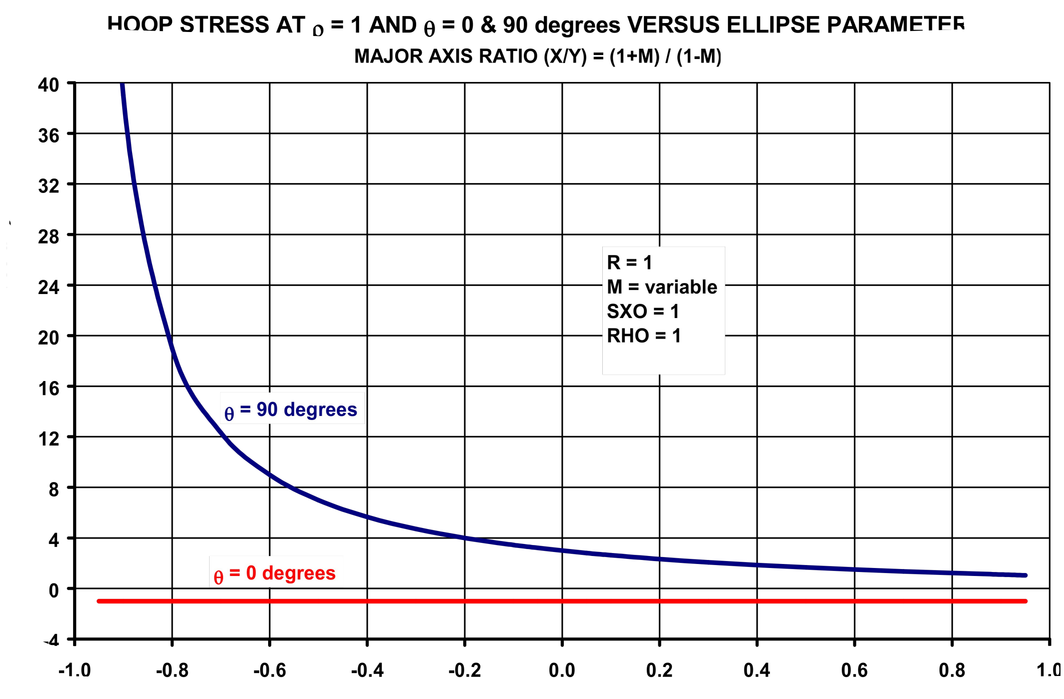
R = 1.0
M = 0.3333
SXO = 10.0
RHO = 1.0
NU = 0.3




RESULTING STRESSES AT HOLE BOUNDARY σξ, ση & τηξ VERSUS ANGULAR COORDINATE
20
15
10
5
0
-5
-10
0
10
20
30
40
50
60
70
80
90
ANGULAR COORDINATE, degrees
σξ
ση, τηξ
R = 1.0
M = 0.3333
SXO = 10.0
RHO = 1.0
NU = 0.3
ση = NORMAL STRESS IN α DIRECTION
σξ = OTHER NORMAL STRESS
τηξ = SHEAR STRESS



The ratio of σ_ξ at the hole boundary ($\rho = 1$) and $\theta = 90^o$ to the uniform tension at infinity is usually called the stress concentration factor. Mechanical design textbooks always give this factor for a circular hole. For the circular hole it is 3.0. The plot below is based on computer runs for different values of the ellipse parameter in the range $-1 < M < 1$ and for unit tension at infinity. When m = 0, the ellipse becomes a circle. The plot shows that the stresses can become large at the boundaries of elliptical holes.



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APPENDIX A – EVALUATION OF INTEGRALS IN EQUATIONS A AND B

$$\oint_\gamma \frac{\varphi(\sigma)\cdot d\sigma}{\sigma-\zeta} + \oint_\gamma \frac{\omega(\sigma)}{\overline{\omega}'(\overline{\sigma})}\cdot\frac{\overline{\varphi}'(\overline{\sigma})\cdot d\sigma}{\sigma-\zeta} + \oint_\gamma \frac{\overline{\psi}(\overline{\sigma})\cdot d\sigma}{\sigma-\zeta} = \oint_\gamma \frac{f(\sigma)\cdot d\sigma}{\sigma-\zeta}$$

A

$$\oint_\gamma \frac{\overline{\varphi}(\overline{\sigma})\cdot d\sigma}{\sigma-\zeta} + \oint_\gamma \frac{\overline{\omega}(\overline{\sigma})}{\omega'(\sigma)}\cdot\frac{\varphi'(\sigma)\cdot d\sigma}{\sigma-\zeta} + \oint_\gamma \frac{\psi(\sigma)\cdot d\sigma}{\sigma-\zeta} = \oint_\gamma \frac{\bar{f}(\overline{\sigma})\cdot d\sigma}{\sigma-\zeta}$$

B

1. Integrals independent of mapping function, $\omega(\zeta)$:

$$\oint_\gamma \frac{\varphi(\sigma)\cdot d\sigma}{\sigma-\zeta} = -2\cdot\pi\cdot i\cdot\varphi(\zeta)$$ $\varphi(\sigma)$ analytic outside of contour

$$\oint_\gamma \frac{\psi(\sigma)\cdot d\sigma}{\sigma-\zeta} = -2\cdot\pi\cdot i\cdot\psi(\zeta)$$ $\psi(\sigma)$ analytic outside of contour

$$\varphi(\zeta) = \frac{a_1}{\zeta} + \frac{a_2}{\zeta^2} + \cdots, \quad \psi(\zeta) = \frac{b_1}{\zeta} + \frac{b_2}{\zeta^2} + \cdots$$ Laurent Expansions

$$\varphi(\sigma) = \frac{a_1}{\sigma} + \frac{a_2}{\sigma^2} + \cdots$$

$$\varphi'(\sigma) = -\frac{a_1}{\sigma^2} - \frac{2\cdot a_2}{\sigma^3} + \cdots, \quad \overline{\varphi}'(\vec{\sigma}) = -\bar{a}_1\cdot\sigma^2 - 2\cdot\bar{a}_2\cdot\sigma^3 + \cdots$$

$$\zeta = \rho\cdot e^{i\cdot\theta}, \ \overline{\zeta} = \rho\cdot e^{-i\cdot\theta}, \ \frac{1}{\overline{\zeta}} = \frac{1}{\rho}\cdot e^{i\cdot\theta}$$ $\rho > 1$ in material region

$$\overline{\varphi}(\overline{\zeta}) = \frac{\bar{a}_1}{\rho}\cdot e^{i\cdot\theta} + \frac{\bar{a}_2}{\rho^2}\cdot e^{2\cdot i\cdot\theta} + \ldots, \quad \overline{\psi}(\overline{\zeta}) = \frac{\bar{a}_1}{\rho}\cdot e^{i\cdot\theta} + \frac{\bar{a}_2}{\rho^2}\cdot e^{2\cdot i\cdot\theta} + \ldots$$

$\overline{\varphi}(\overline{\zeta})$ and $\overline{\psi}(\overline{\zeta})$ are evidently analytic *inside* the γ region so that,

$$\oint_\gamma \frac{\overline{\varphi}(\overline{\zeta})\cdot d\sigma}{\sigma-\zeta} = 0$$

$$\oint_\gamma \frac{\overline{\psi}(\overline{\zeta})\cdot d\sigma}{\sigma-\zeta} = 0$$

2. Integrals for ellipse mapping function

$$z = \omega(\zeta) = R\cdot\left(\zeta + \frac{m}{\zeta}\right),\ \omega(\sigma) = R\cdot\left(\sigma + \frac{m}{\varsigma}\right),\ \overline{\omega}(\overline{\sigma}) = R\cdot\left(\frac{1}{\sigma} + m\cdot\sigma\right)$$

$$\omega'(\zeta) = R\cdot\left(1 - \frac{m}{\zeta^2}\right),\ \overline{\omega}'(\overline{\zeta}) = R\cdot\left(1 - \frac{m}{\overline{\zeta}^2}\right)$$

$$\omega'(\sigma) = R\cdot\left(1 - \frac{m}{\sigma^2}\right),\ \overline{\omega}'(\overline{\sigma}) = R\cdot\left(1 - \frac{m}{\overline{\sigma}^2}\right) = R\cdot\left(1 - m\cdot\sigma^2\right)$$

$$\frac{\omega(\sigma)}{\overline{\omega}'(\overline{\sigma})} = \frac{1}{\sigma}\cdot\frac{\sigma^2+m}{1-m\cdot\sigma^2}\ ,\qquad \frac{\overline{\omega}(\overline{\sigma})}{\omega'(\sigma)} = \sigma\cdot\frac{1+m\cdot\sigma^2}{\sigma^2-m}$$

For Equation A:

$$\oint_\gamma \frac{\sigma^2+m}{1-m\cdot\sigma^2}\cdot\frac{\overline{\varphi}'(\overline{\sigma})\cdot d\sigma}{\sigma\cdot(\sigma-\zeta)} = \oint_\gamma \frac{\sigma^2+m}{1-m\cdot\sigma^2}\cdot\frac{-\overline{a}_1\cdot\sigma - 2\cdot\overline{a}_2\cdot\sigma^2}{\sigma-\zeta}\cdot d\sigma$$

The integrand for the above integral is analytic inside the γ region so the integral vanishes and Equation A gives,

$$\varphi(\zeta) = -\frac{1}{2\cdot\pi\cdot i}\cdot\int_\gamma \frac{f(\sigma)\cdot d\sigma}{\sigma-\zeta}$$

For Equation B:

$$\oint_\gamma \frac{1+m\cdot\sigma^2}{\sigma^2-m}\cdot\frac{\sigma\cdot\varphi'(\sigma)\cdot d\sigma}{\sigma-\zeta} = -2\cdot\pi\cdot i\cdot\frac{1+m\cdot\zeta^2}{\zeta^2-m}\cdot\zeta\cdot\varphi'(\zeta)$$

Since the integrand is analytic in the material region, Equation B becomes,

$$\psi(\zeta) = -\frac{1}{2\cdot\pi\cdot i}\cdot\int_\gamma \frac{\overline{f}(\overline{\sigma})\cdot d\sigma}{\sigma-\zeta} - \frac{1+m\cdot\zeta^2}{\zeta^2-m}\cdot\zeta\cdot\varphi'(\zeta) \qquad \text{(for the circle } m = 0\text{)}$$

APPENDIX B, FORTRAN SOURCE CODE AND ILLUSTRATIVE INPUT FILE FOR PROGRAM MUSK2

SOURCE CODE:

```
C=====================================================================
      PROGRAM MUSK2
C
C   JUNE 26, 2015
C
C   FINDS STRESSES AND DISPLACEMENTS FOR ELLIPTICAL HOLE IN AN
C   INFINITE PLATE WITH UNIFORM AXIAL STRESS IN X-DIRECTION AT
C   INFINITY.
C
C   MEAN SEMI-AXIS LENGTH = R
C
C   X-DIRECTION SEMI-AXIS LENGTH = (1+M)*R
C
C   Y-DIRECTION SEMI-AXIS LENGTH = (1-M)*R
C
C   INPUT DATA:
C
C    R      = MEAN SEMI-AXIS LENGTH
C    M      = ELLIPSE PARAMETER
C    SXO    = UNIFORM TENSILE STRESS AT INFINITY
C    RHO    = RADIUS IN XI-ETA PLANE, = 1 ON HOLE BOUNDARY
C    NU     = POISSON'S RATIO
C
C   OUTPUT DATA:
C
C    TABULATION FOR THETA = 0 TO THETA = 0.5*PI OF,
C
C     THETA  = ANGLE TO POINT IN XI-ETA PLANE, deg
C     X      = COORDINATE IN X-DIRECTION IN X-Y PLANE
C     Y      = COORDINATE IN Y-DIRECTION IN X-Y PLANE
C     ALT    = ANGLE IN Z-PLANE TO NORMAL FROM X-DIRECTION, deg
C     XI     = COORDINATE IN X-DIRECTION IN XI-ETA PLANE
C     ETA    = COORDINATE IN Y-DIRECTION IN XI-ETA PLANE
C     SXI    = EXTRA NORMAL STRESS IN Z-PLANE, NORMAL DIRECTION
C     SETA   = EXTRA NORMAL STRESS IN Z-PLANE, NORMAL DIRECTION
C     TAU    = EXTRA SHEAR STRESS IN Z-PLANE, NORMAL DIRECTION
C     GUXI   = (SHEAR MODULUS)*(DISPLACEMENT IN NORMAL DIRECTION)
C     GUETA  = (SHEAR MODULUS)*(DISPLACEMENT IN NORMAL DIRECTION)
C     STXI   = TOTAL NORMAL STRESS IN Z-PLANE, NORMAL DIRECTION
C     STETA  = TOTAL NORMAL STRESS IN Z-PLANE, NORMAL DIRECTION
C     TAUT   = TOTAL SHEAR STRESS IN Z-PLANE, NORMAL DIRECTION
C     R(PHI)  = REAL(PHI)
C     I(PHI)  = IMAG(PHI)
C     R(PHIP) = REAL(PHIP)
C     I(PHIP) = IMAG(PHIP)
C     R(PHIPP)= REAL(PHIPP)
C     I(PHIPP)= IMAG(PHIPP)
C     R(CHI)  = REAL(CHI)
C     I(CHI)  = IMAG(CHI)
C     R(CHIP) = REAL(CHIP)
```

```
C      I(CHIP) = IMAG(CHIP)
C      ANGZ    = ANGULAR COORDINATE IN Z-PLANE = ATAN(Y/X)
C
      IMPLICIT
     * NONE
C
      REAL
     * R, M, SXO, RHO, NU,
     * PI, RAT, FACU, XI, ETA, ALF, ANGZ, SUM, DIFF, X, Y, RHOR,
     * THETA, SXI, SETA, TAU, GUXI, GUETA, STXI, STETA, TAUT
C
      COMPLEX
     * PSI, Z, PHI, WRT1, WRT2, PHIP, PHIPP, CHI, CHIP, COMB, GU,
     * EALF1, EALF2
C
      INTEGER
     * I
C
      OPEN (2,FILE='MUSK2.INP',STATUS='OLD',ACCESS='SEQUENTIAL',
     *     FORM='FORMATTED')
C
      OPEN (3,FILE='MUSK2.OUT',STATUS='UNKNOWN',ACCESS='SEQUENTIAL',
     *     FORM='FORMATTED')
C
      READ (2,*) R
C
      READ (2,*) M
C
      READ (2,*) SXO
C
      READ (2,*) RHO
C
      READ (2,*) NU
C
      IF (RHO .LT. 1.) THEN
C
       PRINT *,' RHO MUST BE > 1, PROBLEM ABORTED'
C
       STOP
C
      END IF
C
      WRITE (3,1000)
     * R, M, SXO, RHO, NU
C
 1000 FORMAT (
     *1X,'OUTPUT DATA FILE FOR PROGRAM MUSK2, JUNE 23, 2015',//,
     *2X,'INPUT DATA:',//,
     *3X,'R    = MEAN LENGTH FOR SEMI-AXES             = ',G12.4,/,
     *3X,'M    = ELLIPSE PARAMETER                     = ',G12.4,/,
     *3X,'SXO  = X-DIRECTION TENSION AT INFINITY       = ',G12.4,/,
     *3X,'RHO  = RADIUS IN XI, ETA COORDINATES         = ',G12.4,/,
     *3X,'NU   = RATIO OF POISSON                      = ',G12.4,//,
     *3X,'MAJOR AXIS = (1+M)*R, MINOR AXIS = (1-M)*R',//,
     *2X,'OUTPUT DATA:',//,
     *3X,'IN TABULATION BELOW,',//,
```

```
     *4X,'THETA = CW ANGLE FROM XI DIRECTION IN PSI-PLANE, deg',/,
     *4X,'X     = X-COORDINATE IN Z-PLANE',/,
     *4X,'Y     = Y-COORDINATE IN Z-PLANE',/,
     *4X,'ALF   = CW ANGLE OF NORMAL FROM X DIRECT. IN Z-PLANE, deg',/,
     *4X,'XI    = XI-COORDINATE IN PSI-PLANE',/,
     *4X,'ETA   = ETA-COORDINATE IN PSI-PLANE',/,
     *4X,'SXI   = EXTRA NORMAL STRESS IN Z-PLANE, NORMAL DIRECTION',/,
     *4X,'SETA  = EXTRA NORMAL STRESS IN Z-PLANE, NORMAL DIRECTION',/,
     *4X,'TAU   = EXTRA SHEAR STRESS IN Z-PLANE, NORMAL DIRECTION',/,
     *4X,'GUXI  = EXTRA (SHEAR MODULUS)*(XI NORMAL DISPLACEMENT)',/,
     *4X,'GUETA = EXTRA (SHEAR MODULUS)*(ETA NORMAL DISPLACEMENT)',/,
     *4X,'STXI  = TOTAL NORMAL STRESS IN Z-PLANE, NORMAL DIRECTION',/,
     *4X,'STETA = TOTAL NORMAL STRESS IN Z-PLANE, NORMAL DIRECTION',/,
     *4X,'TAUT  = TOTAL SHEAR STRESS IN Z-PLANE, NORMAL DIRECTION',/,
     *4X,'R(PHI)  = REAL(PHI)',/,
     *4X,'I(PHI)  = IMAG(PHI)',/,
     *4X,'R(PHIP) = REAL(PHIP)',/,
     *4X,'I(PHIP) = IMAG(PHIP)',/,
     *4X,'R(PHIPP)= REAL(PHIPP)',/,
     *4X,'I(PHIPP)= IMAG(PHIPP)',/,
     *4X,'R(CHI)  = REAL(CHI)',/,
     *4X,'I(CHI)  = IMAG(CHI)',/,
     *4X,'R(CHIP) = REAL(CHIP)',/,
     *4X,'I(CHIP) = IMAG(CHIP)',/,
     *4X,'ANGZ    = ANGULAR COORDINATE IN Z-PLANE',//,
     *1X,4X,'THETA',4X,6X,'X',6X,6X,'Y',6X,5X,'ALF',5X,6X,'XI',5X,
     *5X,'ETA',5X,5X,'SXI',5X,5X,'SETA',4X,5X,'TAU',5X,
     *5X,'GUXI',4X,4X,'GUETA',4X,5X,'STXI',4X,4X,'STETA',4X,
     *5X,'TAUT',4X,4X,'R(PHI)',3X,4X,'I(PHI)',3X,3X,'R(PHIP)',3X,
     *3X,'I(PHIP)',3X,3X,'R(PHIPP)',2X,3X,'I(PHIPP)',2X,4X,'R(CHI)',3X,
     *4X,'I(CHI)',3X,3X,'R(CHIP)',3X,3X,'I(CHIP)'3X,5X,'ANGZ',/)
C
      PI = 4. * ATAN(1.0)
C
      RAT = ((1. + M)/(1. - M))**2
C
      RHOR = (RHO + M / RHO) / (RHO - M / RHO)
C
      FACU = (3. - NU) / (1. + NU)
C
      DO 100 I = 1, 51
C
C       THETA IS A COORDINATE IN THE PSI-PLANE, PSI = XI + i*ETA
C
       THETA = 0.5 * PI * (I - 1.) / 50.
C
       IF (THETA .GE. 0.5*PI) THEN
C
        XI = 0.
C
        ETA = RHO
C
       ELSE
C
        XI = RHO * COS(THETA)
C
```

```
       ETA = RHO * SIN(THETA)
C
      END IF
C
C      PSI IS COMPLEX COORDINATE IN PSI-PLANE
C
      PSI = CMPLX(XI, ETA)
C
C      Z = COMPLEX COORDINATE IN Z-PLANE, Z = X + i*Y
C      OBTAIN USING CONFORMAL MAPPING FOR AN ELLIPSE
C
      Z = R * PSI + M * R / PSI
C
      X = REAL(Z)
C
      Y = AIMAG(Z)
C
C      ANGZ = ANGULAR COORDINATE IN Z-PLANE
C
      IF (X .EQ. 0.) THEN
C
       ANGZ = 0.5 * PI
C
      ELSE
C
       ANGZ = ATAN(Y / X)
C
      END IF
C
C      ALF = CW ANGLE FROM X-DIRECTION TO NORMAL IN Z-PLANE
C
      IF (ANGZ .LE. 0.) THEN
C
       ALF = 0
C
      ELSE IF (ANGZ .GE. 0.999 * 0.5 * PI) THEN
C
       ALF = 0.5 * PI
C
      ELSE
C
       ALF = ATAN(RAT * TAN(THETA) / RHOR)
C
      END IF
C
C      EALF1 = COMPLEX CW ROTATION OF ALF
C
      EALF1 = CMPLX(COS(ALF), SIN(ALF))
C
C      EALF2 = COMPLEX CW ROTATION OF 2*ALF
C
      EALF2 = CMPLX(COS(2.*ALF), SIN(2.*ALF))
C
C      F'(Z) = F'(PSI) * WRT1
C      F"(Z) = F"(PSI) * (WRT1**2) + F'(PSI) * WRT2
C
```

```
      WRT1 = 1. / (R * (1. - M / (PSI**2)))
C
      WRT2 = -2. * M / ((R**2) * ((PSI - M / PSI)**3))
C
C     GET DERIVATIVES WITH RESPECT TO Z FOR PHI, PHIP AND CHI
C
      PHI = 0.5 * SXO * R * (1. - M) / PSI
C
      PHIP = -0.5 *SXO * R * ((1. - M) / (PSI**2)) * WRT1
C
      PHIPP = (SXO * R * (1. - M) / (PSI**3)) * (WRT1**2)
     *        - 0.5 * SXO * R * ((1. - M) / (PSI**2)) * WRT2
C
      CHI = 0.5 * SXO * R * (1.- M) *
     * ((1. + M * (PSI**2)) / (PSI * ((PSI**2) - M) ) - 1. / PSI)
C
      CHIP = (0.5 *SXO * R * (1. - M) / (PSI**2 * ((PSI**2 - M)**2)))
     *   * (2. * M * (PSI**2) * (PSI**2 - M)
     *   - (3. * (PSI**2) - M) * (1. + M * (PSI**2))
     *   + (PSI**2 - M)**2) * WRT1
C
C      SUM = SXI + SETA
C
      SUM = 4. * REAL(PHIP)
C
C      COMB = SXI - SETA + 2*i*TAU
C
      COMB = 2. * EALF2 * (CONJG(Z) * PHIPP + CHIP)
C
      DIFF = REAL(COMB)
C
C      TAU, SXI & SETA ARE STRESSES TO BE SUPERPOSED ON INITIAL TENSION
C
      TAU = 0.5 * AIMAG(COMB)
C
      SXI = 0.5 * (SUM - DIFF)
C
      SETA = 0.5 * (SUM + DIFF)
C
C      STXI, STETA & TAUT ARE ACTUAL STRESSES AFTER SUPERPOSITION
C
      STXI = SXI + 0.5 * SXO * (1. + COS(2.*ALF))
C
      STETA = SETA + 0.5 * SXO * (1. - COS(2.*ALF))
C
      TAUT = TAU - 0.5 * SXO * SIN(2.*ALF)
C
C      GU = (SHEAR MODULUS)*(UXI + i*UETA), UXI & UETA ARE DISPLACEMENTS
C
      GU = 0.5 * EALF1 * (FACU * CONJG(PHI) - CONJG(Z) * PHIP - CHI)
C
      GUXI = REAL(GU)
C
      GUETA = -AIMAG(GU)
C
      WRITE (3,1100)
```

```
    * THETA*180./PI, REAL(Z), AIMAG(Z), ALF*180./PI, XI, ETA,
    * SXI, SETA, TAU, GUXI, GUETA, STXI, STETA, TAUT,
    * REAL(PHI), AIMAG(PHI), REAL(PHIP), AIMAG(PHIP), REAL(PHIPP),
    * AIMAG(PHIPP), REAL(CHI), AIMAG(CHI), REAL(CHIP), AIMAG(CHIP),
    * ANGZ*180./PI
C
 1100  FORMAT (
    *1X,25(1X,G12.4))
C
  100 CONTINUE
C
      WRITE (3,1200)
C
 1200 FORMAT ( /,
    *1X,4X,'THETA',4X,6X,'X',6X,6X,'Y',6X,5X,'ALF',5X,6X,'XI',5X,
    *5X,'ETA',5X,5X,'SXI',5X,5X,'SETA',4X,5X,'TAU',5X,
    *5X,'GUXI',4X,4X,'GUETA',4X,5X,'STXI',4X,4X,'STETA',4X,
    *5X,'TAUT',4X,4X,'R(PHI)',3X,4X,'I(PHI)',3X,3X,'R(PHIP)',3X,
    *3X,'I(PHIP)',3X,3X,'R(PHIPP)',2X,3X,'I(PHIPP)',2X,4X,'R(CHI)',3X,
    *4X,'I(CHI)',3X,3X,'R(CHIP)',3X,3X,'I(CHIP)'3X,5X,'ANGZ')
C
      END
C=======================================================================
```

ILLUSTRATIVE INPUT DATA FILE:

```
1.     = R   = MEAN LENGTH OF SEMI-AXES
0.3333 = M   = ELLIPSE PARAMETER
10.    = SXO = X-DIRECTION
1.     = RHO = RADIUS IN XI-ETA COORDINATES
0.3    = NU  = POISSON'S RATIO
```

THIS IS FILE: Musk1.doc