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SOME FORMULATIONS COMMONLY USED IN CONTINUUM MECHANICS

by Paul Paslay, P.E. #44278

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I INTRODUCTION

There are two essential components associated with the development of constitutive equations that mathematically represent real materials. The first component is the proper mathematical formulation of the quantities appearing in the constitutive equations. There are physical constraints that are common to all materials that dictate the way these quantities are defined and they may be expressed mathematically. The other component in the development is more in the realm of physics as it is the underlying understanding of the characteristics of a particular material. This work is an attempt to address the considerations associated with the first component of the development of appropriate constitutive equations. The presentation is restricted to Cartesian spatial and reference coordinate systems so the distinction between covariant and contravariant components is not made. In addition, only minimal references to mathematical restrictions on theorems, such as continuity requirements, are made in the text.

After some introductory material the concept of motion is described and the topic of strain is presented in detail. Only two strain definitions are considered in this work. One is referred to the spatial coordinate system while the other is referred to the reference coordinate system. The reader should note that for small strains neither of the tensor strains considered reduce to the usual engineering definition of strain owing to a factor of one half on the tensor shear strains. Following the coverage of strain the issue of strain rate is presented.

The topic of stress is covered next. The restraints on the stress tensor imposed by Newton's First and Second Laws are derived. Although stress is closely associated with a spatial coordinate system, the formulation for stress is presented in both the spatial and reference coordinates. The coverage of stress is followed by a presentation of four definitions of stress rate. The stress rate presentation is rather detailed as the author feels this topic is often not covered or is poorly covered in some textbooks in Continuum Mechanics.

The final section considers the relationship of constitutive equations to Classical Thermodynamics. The section presents a set of elementary constitutive equations for several materials in thermal equilibrium. Expressions are derived for the internal energy, heat flow rate and entropy production rate. These derivations are based on the requirement that a material in thermal equilibrium must have entropy that depends only on the state of the material.

The appendix is a review of the usual, elementary, Classical Thermodynamics formulation.

II PRELIMINARY CONSIDERATIONS

The change of configuration of a material body is described in this work using a reference and a current set of coordinates. Both of these coordinate systems are three-dimensional, Cartesian, right-handed coordinate systems. Only the current set of coordinates is considered in this section.

The current set of coordinates is the one in which evaluation of the quantities studied are made. It is a spatial coordinate system in an inertial space, one without acceleration. Each set of coordinates x_1, x_2, and x_3 (referred to as **x**) locate a fixed point in the inertial space, i.e. $\vec{x} = x_1 \cdot \vec{v}_1 + x_2 \cdot \vec{v}_2 + x_3 \cdot \vec{v}_3$. The distance, ds, between two points separated by the differential amount d**x,** i.e. $d\vec{x} = dx_1 \cdot \vec{v}_1 + dx_2 \cdot \vec{v}_2 + dx_3 \cdot \vec{v}_3$ is,

$$ds = \sqrt{dx_1^{\ 2} + dx_2^{\ 2} + dx_3^{\ 2}}$$

The base vectors $\vec{v}_1, \vec{v}_2$ and $\vec{v}_3$ (in most texts called $\vec{i}$, $\vec{j}$ and $\vec{k}$) relate the vector $d\vec{s}$ joining points **x** and **x**+d**x** to d**x** as,

$$d\vec{s} = dx_1 \cdot \vec{v}_1 + dx_2 \cdot \vec{v}_2 + dx_3 \cdot \vec{v}_3$$

In the following, unless otherwise specified, summation is implied by repeated, subscripted, letter (not numbered) indices in a term so the above two equations may be written as,

$$ds = \sqrt{dx_i \cdot dx_i}$$

$$d\vec{s} = dx_i \cdot \vec{v}_i$$

Any vector, $\vec{a}$, may be expressed in terms of its components, a_1, a_2 and a_3 as,

$$.\vec{a} = a_i \cdot \vec{v}_i$$

When a second inertial, spatial coordinate system with base vectors, $\vec{\bar{v}}_1, \vec{\bar{v}}_2$ and $\vec{\bar{v}}_3$ is introduced the vector may be expressed in either coordinate system as,

$$.\vec{a} = a_i \cdot \vec{v}_i = \bar{a}_i \cdot \vec{\bar{v}}_i$$

The dot product of this equation with each of the base vectors leads to the following results,

$$\bar{a}_j = a_i \cdot \vec{v}_i \bullet \vec{\bar{v}}_j \equiv c_{ji} \cdot a_i$$

$$a_j = \bar{a}_i \cdot \vec{v}_j \bullet \vec{\bar{v}}_i \equiv \bar{c}_{ji} \cdot \bar{a}_i$$

where,

$$c_{ji} = \vec{v}_i \bullet \vec{\bar{v}}_j$$

$$\bar{c}_{ji} = \vec{v}_j \bullet \vec{\bar{v}}_i$$

The matrices associated with the coefficients, $[c]$ and $[\bar{c}]$, are both orthogonal matrices so that,

$[c] = [\bar{c}]^{-1} = [\bar{c}]^{T}$ and the determinant of $[c] = +1$

For the differential length vector d**x**,

$$d\bar{x}_j = c_{ji} \cdot dx_i \quad \text{and} \quad dx_j = \bar{c}_{ji} \cdot d\bar{x}_i$$

so that

$$c_{ji} = \frac{\partial \bar{x}_j}{\partial x_i} \quad \text{and} \quad \bar{c}_{ji} = \frac{\partial x_j}{\partial \bar{x}_i}$$

The determinant of a matrix formed from a_{ij} is denoted as $\|a\|$ and may be evaluated using,

$$\|a\| = \left\| \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \right\| = e_{ijk} \cdot a_{i1} \cdot a_{j2} \cdot a_{k3}$$

where e_{ijk} is the Levi-Chivita symbol defined by,

$e_{123} = e_{312} = e_{231} = +1$ and $e_{132} = e_{321} = e_{213} = -1$

otherwise $e_{ijk} = 0$

The cross product of two vectors $\vec{a}$ and $\vec{b}$ may be determined by,

$$\vec{a} \times \vec{b} = e_{ijk} \cdot \vec{v}_i \cdot a_j \cdot b_k$$

and the volume, VO, determined by the three vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$, is,

$$VO = \vec{a} \times \vec{b} \bullet \vec{c} = e_{ijk} \cdot \|H\|$$

where,

$$[H] = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$$

Also note that,

$$e_{lik} \cdot \frac{\partial x_i}{\partial \bar{x}_l} \cdot \frac{\partial x_j}{\partial \bar{x}_m} \cdot \frac{\partial x_k}{\partial \bar{x}_n} = \left\| \frac{\partial x}{\partial \bar{x}} \right\| \cdot e_{lmn} = e_{lmn}$$

and recall that,

$e_{ijk} = -e_{jik}$ $\quad e_{ijk} = -e_{ikj}$ $\quad e_{ijk} = -e_{kji}$

The coordinate transformations above show that these are tensor transformations. Consider a second rank tensor ξ_{ij} (like stress or strain) that undergoes a change of coordinates to obtain $\bar{\xi}_{ij}$. The transformation is given by,

$$\bar{\xi}_{ij} = c_{ik} \cdot c_{jl} \cdot \xi_{kl}$$

If i = j then,

$$\bar{\xi}_{ii} = c_{ik} \cdot c_{il} \cdot \xi_{kl} = \delta_{kl} \cdot \xi_{kl} = \xi_{kk}$$

where δ_{ij} is the Kronecker Delta (δ_{ij} = 1 if i = j; = 0 otherwise). . This last equation shows that the sum of the diagonal terms (the trace) of the [ξ] matrix is unchanged by the change of coordinates. Therefore ξ_{ii} is called an invariant of ξ_{ij}. It is straightforward to show other combinations that are invariant in a change of coordinates. The three simplest for a second rank tensor are given below.

I1 $\quad = \xi_{ii} \quad = Tr[\xi]$

I2 $\quad = \xi_{ij} \cdot \xi_{ji} \quad = Tr\big[[\xi] \cdot [\xi]\big] \equiv Tr[\xi]^2$

$$I3 \quad = \xi_{ij} \cdot \xi_{jk} \cdot \xi_{ki} = \mathrm{Tr}\left([\xi]\cdot[\xi]\cdot[\xi]\right) \equiv \mathrm{Tr}\left([\xi]^3\right)$$

The well known Cayley-Hamilton Theorem delimits the number of independent invariants. This theorem states that a square matrix satisfies its characteristic equation. The characteristic equation for $[\xi]$ is,

$$\left\| [\xi] - \lambda \cdot [I] \right\| = 0$$

or,

$$\|\xi\| + \left(-\xi_{22}\cdot\xi_{33} - \xi_{33}\cdot\xi_{11} - \xi_{11}\cdot\xi_{22} + \xi_{31}\cdot\xi_{13} + \xi_{32}\cdot\xi_{23} + \xi_{12}\cdot\xi_{21}\right)\cdot\lambda + \left(\xi_{11} + \xi_{22} + \xi_{33}\right)\cdot\lambda^2 + \lambda^3 = 0$$

and upon applying the theorem and introducing I1, I2 and I3,

$$-[\xi]^3 + I1\cdot[\xi]^2 - \left(-\tfrac{1}{2}\cdot I2 + \tfrac{1}{2}\cdot I1^2\right)\cdot[\xi] + \left(\tfrac{1}{6}\cdot I1^3 - \tfrac{1}{2}\cdot I2\cdot I1 + \tfrac{1}{3}\cdot I3\right)\cdot[I] = 0$$

Therefore, all invariants of the form $\mathrm{Tr}\left([\xi]^n\right)$ where n is an integer = 3 may be expressed in terms of I1, I2 and I3. Consequently there are only three independent invariants of this form. A second set of invariants that are commonly used are K1, K2 and K3 and they are related to I1, I2 and I3 as follows,

$$K1 = I1$$

$$K2 = -\tfrac{1}{2}\cdot I2 + \tfrac{1}{2}\cdot I1^2$$

$$K3 = \tfrac{1}{6}\cdot I1^3 - \tfrac{1}{2}\cdot I2\cdot I1 + \tfrac{1}{3}\cdot I3$$

Gauss' theorem of the gradient is used in the following developments. In terms of the notation used here it is,

$$\int\int_{\text{volume}}\int \frac{\partial f}{\partial x_i}\cdot d(\text{volume}) = \int\int_{\text{surface}} f\cdot n_i \cdot d(\text{surface})$$

where f is a function of position in the volume and $\vec{n}$ is the outward, normal, unit length vector on the surface.

III STRAIN

III.1 DESCRIPTION OF MOTION

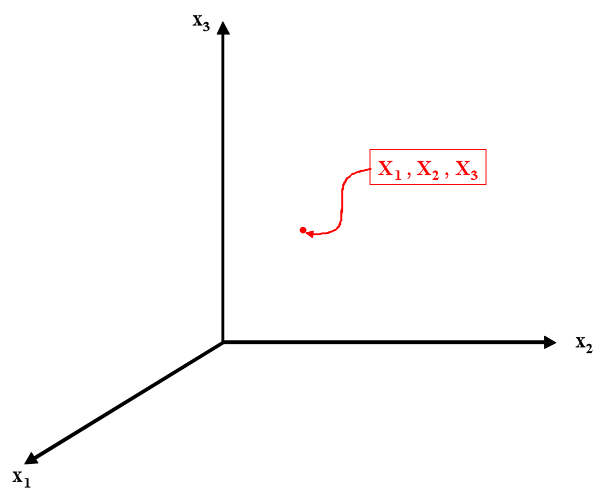
In preparation for defining strain a rigid body motion is defined first. For a rigid body motion the straight line distance between any two material points in the same body is unchanged by the motion. Any other motion has strain occurring in at least part of the body.

In this work two distinct coordinate systems are used to define motion. They are,

1. The material coordinate system **X** with components X_α that define constant mass material points fixed in the body and have base vectors $\vec{V}_\alpha$. These material points are referred to as particles in the following development. This is referred to as the Lagrangian description.

2. The spatial coordinate system **x** with components x_i that define points in an inertial space and have base vectors $\vec{v}_i$.

The sketch below illustrates the systems.



In order to simplify subsequent work the initial values for **x** are assumed to equal the reference coordinates **X**. That is, for the particle **X**,

Initial value of x_1 before motion occurs = X_1
Initial value of x_2 before motion occurs = X_2
Initial value of x_3 before motion occurs = X_3

and in this initial state the strain in the body is zero everywhere in the body. When motion of the body occurs, the particle **X** has a change of its spatial coordinates, **x**. In this section the strain is determined for the change in configuration from the initial state to the current configuration defined to be at the end of the motion. This change is given by **X**(**x**) in the current configuration. In order to help make the development tractable subscripts in the current configuration are Latin symbols while those in the material coordinates are Greek symbols. Consider two particles **X** and **X** + d**X** separated by a differential distance, dS, in the initial, reference configuration. The relation of dS to d**X** is given by,

$$dS^2 = dX_\alpha \cdot dX_\alpha$$

and the vector joining **X** and **X** + d**X** is given by,

$$d\vec{S} = dX_\alpha \cdot \vec{V}_\alpha$$

where $\vec{V}_\alpha$ are the base vectors associated with the X_α coordinates.

As a result of the motion the distance between **X** and **X** + d**X** has changed from its initial configuration value. This changed distance may be found using the **X**(**x**) relationship. Note that the relationship may always be inverted to determine **x**(**X**) since contacting particles cannot be allowed to come out of contact nor can we allow two particles to occupy the same spatial point. This is a fundamental concept when describing a classical continuous body. As a result,

$$dX_\alpha = \frac{\partial X_\alpha}{\partial x_i} \cdot dx_i$$

$$dx_i = \frac{\partial x_i}{\partial X_\alpha} \cdot dX_\alpha$$

and

$$\frac{\partial X_\alpha}{\partial x_i} \cdot \frac{\partial x_i}{\partial X_\beta} = \delta_{\alpha\beta}$$

$$\frac{\partial x_i}{\partial X_\alpha} \cdot \frac{\partial X_\alpha}{\partial x_j} = \delta_{ij}$$

The equations above show that the two matrices for the partial derivatives are inverses of each other.

A differential length vector $\vec{V}_\alpha \cdot dX_\alpha$ in the reference configuration becomes the vector $\vec{\mathbf{V}}_\alpha \cdot dX_\alpha$ in the current configuration where,

$$\vec{\mathbf{V}}_{\alpha} = \frac{\partial x_i}{\partial X_{\alpha}} \cdot \vec{v}_i$$

This vector is referred to as the Chapman distorted base vector in this work.

Strain that occurs in a body going from the reference configuration to the current configuration may be evaluated mathematically for a point by determining,

$$ds^2 - dS^2 = dx_i \cdot dx_i - dX_{\alpha} \cdot dX_{\alpha}$$

III.2 SPATIAL STRAIN TENSOR

III.2.1 FORMULATION

The first strain tensor to be considered is referred to as the Almansi-Hamel or spatial strain (e_{ij}). It is deduced from the last equation in Section III.1 and

$$dX_\alpha = \frac{\partial X_\alpha}{\partial x_i} \cdot dx_i$$

to obtain,

$$ds^2 - dS^2 = \left(\delta_{ij} - \frac{\partial X_\alpha}{\partial x_i} \cdot \frac{\partial X_\alpha}{\partial x_j}\right) \cdot dx_i \cdot dx_i \equiv 2 \cdot e_{ij} \cdot dx_i \cdot dx_j$$

so that,

$$e_{ij} = \frac{1}{2} \cdot \left(\delta_{ij} - \frac{\partial X_\alpha}{\partial x_i} \cdot \frac{\partial X_\alpha}{\partial x_j}\right)$$

Note that e_{ij} is symmetric. The expression $e_{ij} \cdot dx_i \cdot dx_j$ will not vanish for all, arbitrary, non-zero values of d**x** unless every component of e_{ij} vanishes. Consequently, a rigid body motion from the reference configuration to the current configuration will have all the e_{ij} components vanish. When all the components of e_{ij} do not vanish, a rigid body motion between the reference and current configuration is not possible. Therefore, e_{ij} is a physically acceptable definition of strain. In matrix notation,

$$[e_{ij}] = \begin{bmatrix} e_{11} & e_{12} & e_{13} \\ e_{21} & e_{22} & e_{23} \\ e_{31} & e_{32} & e_{33} \end{bmatrix} = \frac{1}{2} \cdot \left[[I] - \left[\frac{\partial X_\alpha}{\partial x_i}\right]^T \cdot \left[\frac{\partial X_\beta}{\partial x_j}\right] \right]$$

Since $[e_{ij}]$ is real and symmetric, it is always possible to find a change of spatial coordinates that transform $[e_{ij}]$ to the form,

$$[e_I] \equiv \begin{bmatrix} e_I & 0 & 0 \\ 0 & e_{II} & 0 \\ 0 & 0 & e_{III} \end{bmatrix}$$

and e_I, e_{II} and e_{III} are called the principal spatial strains. The axes of the new spatial coordinates are called the principal directions.

An element aligned with the i[th] principle direction that was one unit long in the reference configuration has length $1 + \delta_i$ in the deformed configuration and,

$$e_i = \frac{1}{2} \cdot \left(1 - \frac{1}{(1+\delta_i)^2}\right) \qquad \rightarrow \qquad \delta_i = \frac{1}{\sqrt{1 - 2 \cdot e_i}} - 1$$

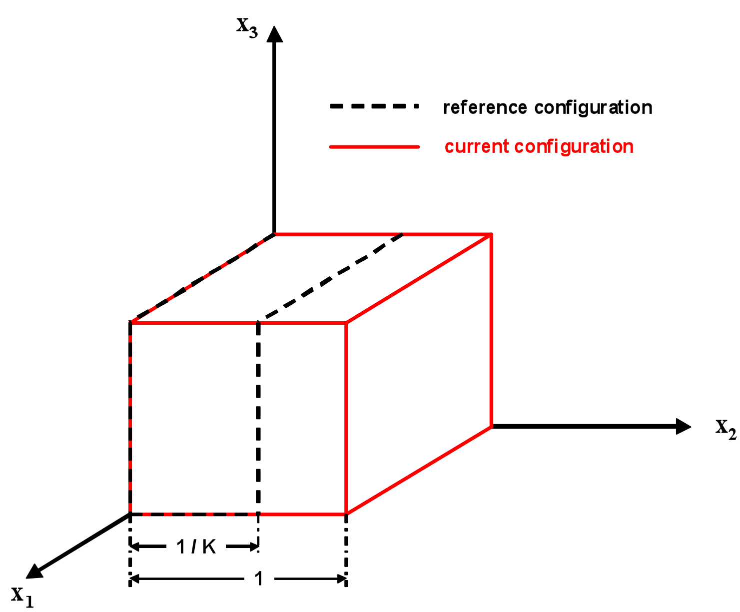
On physical grounds, $-1 < \delta_i < \infty$ so that $-\infty < e_i < \frac{1}{2}$ and this range applies to all the principal strains e_I, e_{II} and e_{III}. δ_i is referred to as a principal spatial extension.

III.2.2 ILLUSTRATIVE EXAMPLES

Uniaxial Strain:

$$\begin{aligned} x_1 &= K \cdot X_1 \\ x_2 &= X_2 \\ x_3 &= X_3 \end{aligned} \qquad \text{or} \qquad \begin{aligned} X_1 &= \frac{x_1}{K} \\ X_2 &= x_2 \\ X_3 &= x_3 \end{aligned} \qquad \text{so that} \quad [e_{ij}] \equiv \begin{bmatrix} \frac{1}{2} \cdot \left(1 - \frac{1}{K^2}\right) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

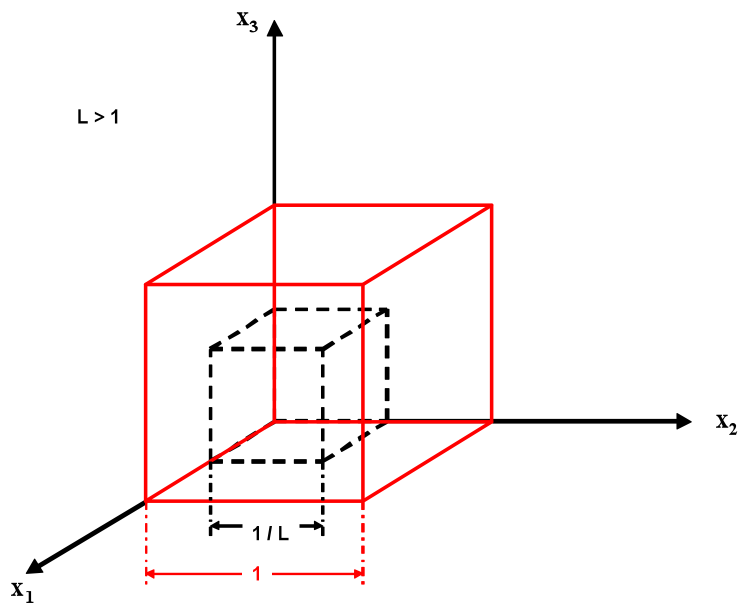
Since the off-diagonal terms .in $[e_{ij}]$ vanish, the three principal spatial strains are $\frac{1}{2} \cdot \left(1 - \frac{1}{K^2}\right)$, 0 and 0. A graphical representation of this uniform strain is shown in the sketch of rectangular parallelepiped shapes below.



Isotropic Strain:

$$x_1 = L \cdot X_1 \qquad X_1 = \frac{x_1}{L}$$
$$x_2 = L \cdot X_2 \quad \text{or} \quad X_2 = \frac{x_2}{L} \quad \text{so that} \quad [e_{ij}] \equiv \begin{bmatrix} \frac{1}{2}\cdot\left(1-\frac{1}{L^2}\right) & 0 & 0 \\ 0 & \frac{1}{2}\cdot\left(1-\frac{1}{L^2}\right) & 0 \\ 0 & 0 & \frac{1}{2}\cdot\left(1-\frac{1}{L^2}\right) \end{bmatrix}$$
$$x_3 = 1 \cdot X_3 \qquad X_3 = \frac{x_3}{L}$$

Since the off-diagonal terms .in $[e_{ij}]$ vanish, the three principal spatial strains are $\frac{1}{2}\cdot\left(1-\frac{1}{L^2}\right)$, $\frac{1}{2}\cdot\left(1-\frac{1}{L^2}\right)$ and $\frac{1}{2}\cdot\left(1-\frac{1}{L^2}\right)$. The sketch below of the transparent cubes illustrates this uniform expansion with the current configuration shown by the solid, red line..



Shear Strain:

$$x_1 = X_1 + M \cdot X_2 \qquad X_1 = x_1 - M \cdot x_2$$
$$x_2 = X_2 \quad \text{or} \quad X_2 = x_2 \quad \text{so that} \quad [e_{ij}] \equiv \begin{bmatrix} 0 & \frac{1}{2}\cdot M & 0 \\ \frac{1}{2}\cdot M & -\frac{1}{2}\cdot M^2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$
$$x_3 = X_3 \qquad X_3 = x_3$$

Note the one-half factor on the off-diagonal terms. For the engineering strains defined in virtually all Strength of Materials textbooks the factor is one rather than one half. The above form with the one half is the tensor definition of strain. In addition, the presence of non-vanishing off-diagonal terms indicates the spatial coordinate system is not aligned with the principal directions. The characteristic equation for the $[e_{ij}]$ matrix is,

$$-\lambda^3 - \tfrac{1}{2}\cdot M^2\cdot\lambda^2 + \tfrac{1}{4}\cdot M^2\cdot\lambda = 0$$

and its roots are the principal strains giving,

$$e_I = -\tfrac{1}{4}\cdot M^2 + \tfrac{1}{4}\cdot\sqrt{M^4 + 4\cdot M^2}$$
$$e_{II} = -\tfrac{1}{4}\cdot M^2 - \tfrac{1}{4}\cdot\sqrt{M^4 + 4\cdot M^2}$$
$$e_{III} = 0$$

and the corresponding principal directions are,

$$\vec{v}_I = \frac{1}{\sqrt{2 + \frac{1}{2}\cdot M^2 - \frac{1}{2}\cdot M\cdot\sqrt{M^2+4}}}\cdot\vec{v}_1 + \frac{-\frac{1}{2}\cdot M + \frac{1}{2}\cdot\sqrt{M^2+4}}{\sqrt{2 + \frac{1}{2}\cdot M^2 - \frac{1}{2}\cdot M\cdot\sqrt{M^2+4}}}\cdot\vec{v}_2$$

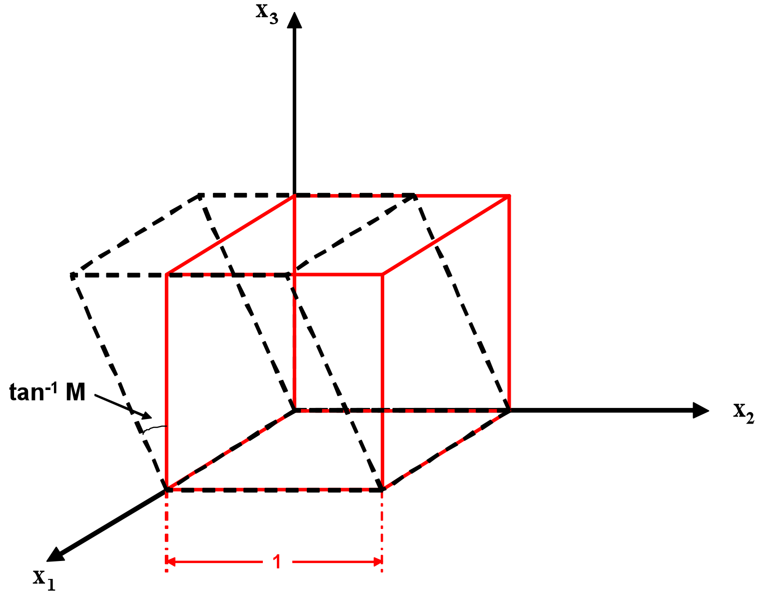
$$\vec{v}_{II} = \frac{\frac{1}{2}\cdot M - \frac{1}{2}\cdot\sqrt{M^2+4}}{\sqrt{2 + \frac{1}{2}\cdot M^2 - \frac{1}{2}\cdot M\cdot\sqrt{M^2+4}}}\cdot\vec{v}_1 + \frac{1}{\sqrt{2 + \frac{1}{2}\cdot M^2 - \frac{1}{2}\cdot M\cdot\sqrt{M^2+4}}}\cdot\vec{v}_2$$

$$\vec{v}_{III} = \vec{v}_3$$

The inclination, γ, of the $\vec{v}_I$ vector to the $\vec{v}_1$ vector is,

$$\gamma = \tan^{-1}\left(-\tfrac{1}{2}\cdot M + \tfrac{1}{2}\cdot\sqrt{M^2+4}\right)$$

This shearing motion is illustrated in the sketch below. The red solid line shows the reference configuration.



x3
x2
x1
tan-1 M
1

III.3 MATERIAL STRAIN TENSOR

III.3.1 FORMULATION

A second strain definition is also based on,

$$ds^2 - dS^2 = dx_i \cdot dx_i - dX_\alpha \cdot dX_\alpha$$

In this case the Green-St.Venant or material strain tensor, $E_{\alpha\beta}$, is found by combining this equation with,

$$dx_i = \frac{\partial x_i}{\partial X_\alpha} \cdot dX_\alpha$$

to obtain,

$$ds^2 - dS^2 = \left(\frac{\partial x_i}{\partial X_\alpha} \cdot \frac{\partial x_i}{\partial X_\beta} - \delta_{\alpha\beta} \right) \cdot dX_\alpha \cdot dX_\alpha$$

From this last equation, an appropriate measure of strain is seen to be,

$$E_{\alpha\beta} = \frac{1}{2} \cdot \left(\frac{\partial x_i}{\partial X_\alpha} \cdot \frac{\partial x_i}{\partial X_\beta} - \delta_{\alpha\beta} \right)$$

Note that $E_{\alpha\beta}$ is symmetric and that in matrix notation,

$$\left[E_{\alpha\beta} \right] = \begin{bmatrix} E_{11} & E_{12} & E_{13} \\ E_{21} & E_{22} & E_{23} \\ E_{31} & E_{32} & E_{33} \end{bmatrix} = \frac{1}{2} \cdot \left[\left[\frac{\partial x_i}{\partial X\alpha} \right]^T \cdot \left[\frac{\partial x_i}{\partial X\beta} \right] - [I] \right]$$

so that principal material strain values and directions may be found. An element parallel to the K^{th} principal direction of unit length in the reference configuration has a length of 1 + Δ_K after straining and the corresponding principal strain, E_K is,

$$E_K = \tfrac{1}{2} \cdot \left((1 + \Delta_K)^2 - 1 \right) \qquad \text{or} \qquad \Delta_K = \sqrt{1 + 2 \cdot E_K} - 1$$

Since, $-1 < \Delta_K < \infty$,

$$-\tfrac{1}{2} < E_K < \infty$$

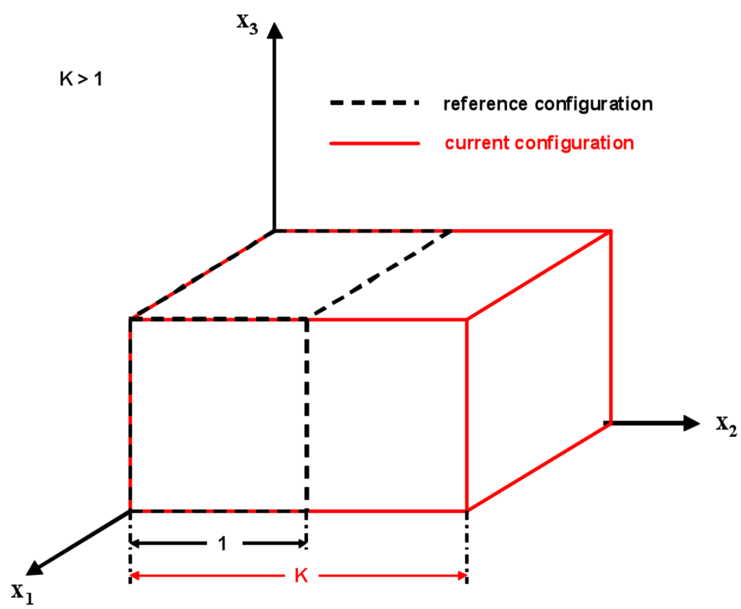
and this is true for all the principal strains. Δ_K is called a principal material extension.

III.3.2 ILLUSTRATIVE EXAMPLES

Uniaxial Strain:

$$x_1 = K \cdot X_1 \qquad X_1 = \frac{x_1}{K}$$
$$x_2 = X_2 \quad \text{or} \quad X_2 = x_2 \quad \text{so that} \quad [E_{\alpha\beta}] \equiv \begin{bmatrix} \frac{1}{2} \cdot (K^2 - 1) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$
$$x_3 = X_3 \qquad X_3 = x_3$$

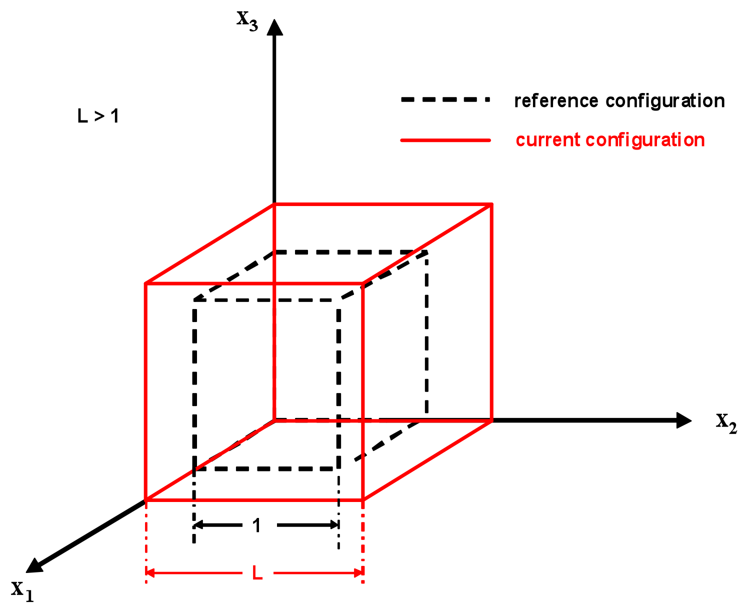
Since the off-diagonal terms .in $[e_{ij}]$ vanish, the principal spatial strains are $\frac{1}{2} \cdot \left(1 - \frac{1}{K^2}\right)$, 0 and 0. A graphical representation of this uniform strain is shown in the sketch of rectangular parallelepiped shapes below.



Isotropic Strain:

$$x_1 = L \cdot X_1 \qquad X_1 = \frac{x_1}{L}$$
$$x_2 = L \cdot X_2 \quad \text{or} \quad X_2 = \frac{x_2}{L} \quad \text{so that} \quad [E_{\alpha\beta}] \equiv \tfrac{1}{2} \cdot (L^2 - 1) \cdot [I]$$
$$x_3 = L \cdot X_3 \qquad X_3 = \frac{x_3}{L}$$

Since the off-diagonal terms .in $[E_{\alpha\beta}]$ vanish, the principal spatial strains are $\frac{1}{2}\cdot(L^2-1)$, $\frac{1}{2}\cdot(L^2-1)$ and $\frac{1}{2}\cdot(L^2-1)$. The sketch below of the transparent cubes illustrates this uniform expansion.



Shear Strain:

$$x_1 = X_1 + M\cdot X_2 \qquad X_1 = x_1 - M\cdot x_2$$
$$x_2 = X_2 \qquad \text{or} \qquad X_2 = x_2 \qquad \text{so that} \quad [E_{\alpha\beta}] \equiv \begin{bmatrix} 0 & \frac{1}{2}\cdot M & 0 \\ \frac{1}{2}\cdot M & \frac{1}{2}\cdot M^2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$
$$x_3 = X_3 \qquad X_3 = x_3$$

Note once more the one-half factor on the off-diagonal terms. For the engineering strains defined in virtually all Strength of Materials textbooks the factor is one rather than one half. In addition, the presence of non-vanishing off-diagonal terms indicates the reference coordinate system is not aligned with the principal directions. The characteristic equation for the $[E_{\alpha\beta}]$ matrix is,

$$-\lambda^3 + \frac{1}{2}\cdot M^2\cdot\lambda^2 + \frac{1}{4}\cdot M^2\cdot\lambda = 0$$

and its roots are the principal strains giving,

$$e_I = \frac{1}{4}\cdot M^2 + \frac{1}{4}\cdot\sqrt{M^4 + 4\cdot M^2}$$
$$e_{II} = \frac{1}{4}\cdot M^2 - \frac{1}{4}\cdot\sqrt{M^4 + 4\cdot M^2}$$
$$e_{III} = 0$$

and the corresponding principal directions are,

$$\vec{V}_I = \frac{1}{\sqrt{2 + \frac{1}{2}\cdot M^2 + \frac{1}{2}\cdot M\cdot\sqrt{M^2+4}}}\cdot\vec{V}_1 + \frac{\frac{1}{2}\cdot M + \frac{1}{2}\cdot\sqrt{M^2+4}}{\sqrt{2 + \frac{1}{2}\cdot M^2 + \frac{1}{2}\cdot M\cdot\sqrt{M^2+4}}}\cdot\vec{V}_2$$

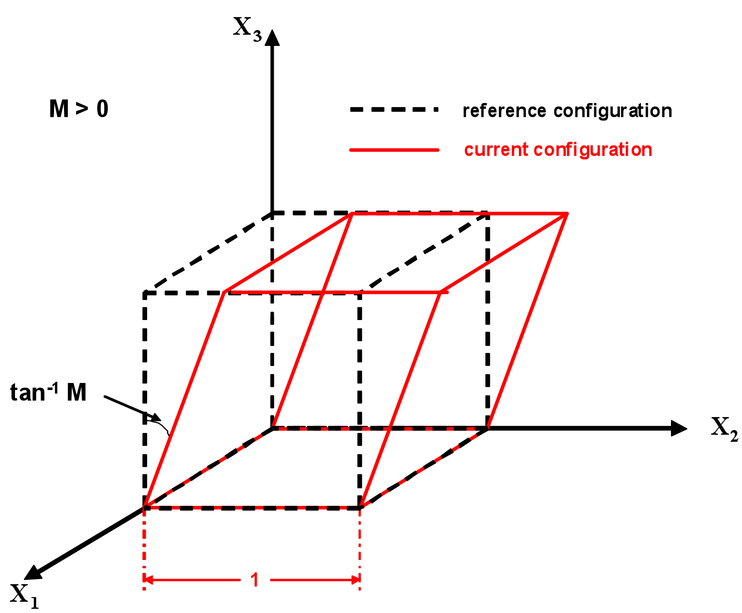
$$\vec{V}_{II} = \frac{-\frac{1}{2}\cdot M - \frac{1}{2}\cdot\sqrt{M^2+4}}{\sqrt{2 + \frac{1}{2}\cdot M^2 + \frac{1}{2}\cdot M\cdot\sqrt{M^2+4}}}\cdot\vec{V}_1 + \frac{1}{\sqrt{2 + \frac{1}{2}\cdot M^2 + \frac{1}{2}\cdot M\cdot\sqrt{M^2+4}}}\cdot\vec{V}_2$$

$$\vec{V}_{III} = \vec{V}_3$$

The inclination, Γ, of the $\vec{V}_I$ vector to the $\vec{V}_1$ vector is given by,

$$\Gamma = \tan^{-1}\left(\frac{1}{2}\cdot M + \frac{1}{2}\cdot\sqrt{M^2+4}\right)$$

This shearing motion is illustrated in the sketch below



III.4 COMPARISON OF SPATIAL AND MATERIAL STRAINS

The formal relations between $[E_{\alpha\beta}]$ and $[e_{ij}]$ are,

$$E_{\alpha\beta} = \frac{\partial x_i}{\partial X_\alpha} \cdot e_{ij} \cdot \frac{\partial x_j}{\partial X_\beta}$$

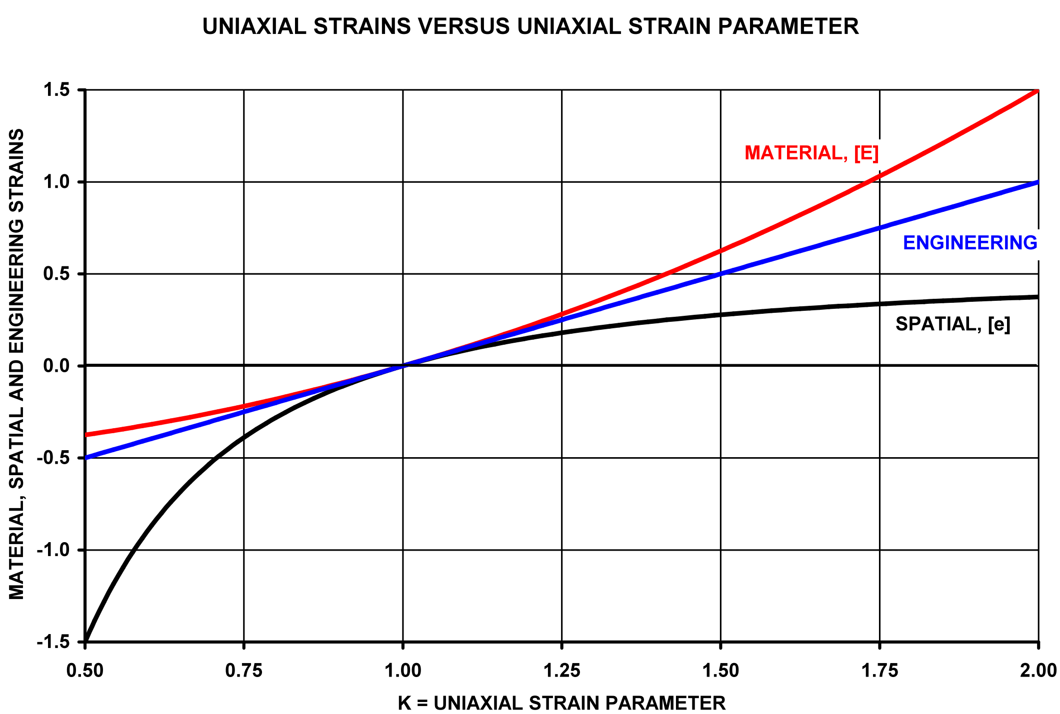
and

$$e_{ij} = \frac{\partial X_\alpha}{\partial x_i} \cdot E_{\alpha\beta} \cdot \frac{\partial X_\beta}{\partial x_j}$$

The case of uniaxial strain in the x_1 direction determined the E_{11} and e_{11} strains as,

$$E_{11} = \tfrac{1}{2} \cdot \left(K^2 - 1\right)$$
$$e_{11} = \tfrac{1}{2} \cdot \left(1 - \frac{1}{K^2}\right)$$

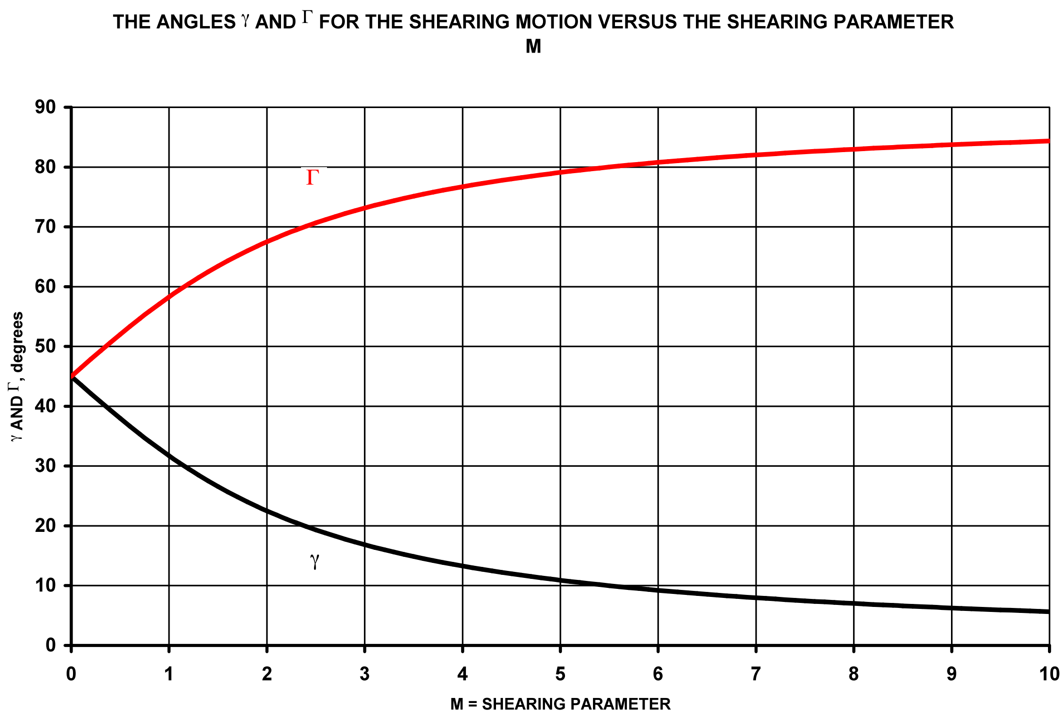
where K is the uniaxial straining parameter. The spatial and material strains are compared for the uniaxial case in the plot below. In addition the usual engineering strain is shown.



The studies of shearing motion for the spatial and materials determined two angles that are,

γ = inclination of first principal direction for the spatial strains to the x_1 axis
Γ = inclination of first principal direction for the material strains to the x_1 axis

The curve below shows how each angle changes as the shearing parameter, M, varies.



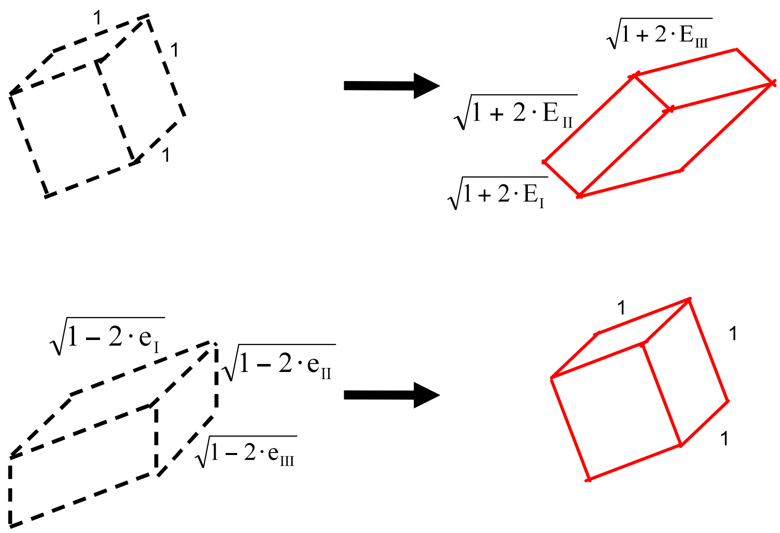
There is no shear when M = 0. Note that for a small amount of shear (M << 1) both angles are close to 45°. The angle for the usual small strain theory is 45°. Relationships between γ and Γ are,

$$\tan(\gamma)\cdot\tan(\Gamma) = 1 \qquad \text{and} \qquad \gamma + \Gamma = 90^\circ$$

In the general case of straining the principal strain magnitudes and directions can be used to identify orientations that transform during straining from an initial rectangular parallelepiped to another rectangular parallelepiped. This result can be anticipated on physical grounds as the initial configuration and the final configuration in the principal strain directions both have vanishing shear strains. The sketches below show the results.

ALL FIGURES ARE RECTANGULAR PARALLELEPIPEDS

INITIAL CONFIGURATION | **FINAL CONFIGURATION**



EDGES ALINED WITH $E_{\alpha\beta}$ PRINCIPAL STRAIN DIRECTIONS SYMBOLS INDCATE LENGTH

EDGES ALINED WITH e_{IJ} PRINCIPAL STRAIN DIRECTIONS SYMBOLS INDCATE LENGTH

The above figure demonstrates that,

$$(1+2\cdot E_P)\cdot(1-2\cdot e_P)=1 \qquad P = \mathrm{I, II, III}$$

so that,

$$E_I = \frac{e_I}{1-2\cdot e_I}$$

$$E_{II} = \frac{e_{II}}{1-2\cdot e_{II}}$$

$$E_{III} = \frac{e_{III}}{1-2\cdot e_{III}}$$

$$e_I = \frac{E_I}{1+2\cdot E_I}$$

$$e_{II} = \frac{E_{II}}{1+2\cdot E_{II}}$$

$$e_{III} = \frac{E_{III}}{1+2\cdot E_{III}}$$

Consequently, when one set of principal strains is known, the other set can be determined using the above equations. Note that the first two equations of this section show that, in general, when either [**E**] or [**e**] is specified, the other strain cannot be found.

Recall the definitions of the principal extensions,

$$\delta_i = \frac{1}{\sqrt{1 - 2 \cdot e_i}} - 1 \qquad\qquad \Delta_K = \sqrt{1 + 2 \cdot E_K} - 1$$

When these definitions are introduced into the above equations relating the spatial and material principal strains there results,

$$\delta_P = \Delta_P \qquad P = I, II, III$$

Finally, the results in this section show, when the components of either [**E**] or [**e**] are all << 1, the spatial and material strains are essentially the same. In this case, the normal strains are approximately equal to the normal engineering strains while there is a one-half factor on the engineering shear strains. Because of the one-half factor the engineering strain is not, strictly speaking, a tensor.

III.5 TIME DERIVATIVES

Experiments on certain real materials demonstrate that under constant load the material response is to have a continually increasing strain. The most common materials exhibiting this behavior are viscous fluids. Since one of the goals of Continuum Mechanics is to develop mathematical means describing and predicting the behavior of real materials a means is needed to describe fluid-like behavior. The simplest physical concept for a fluid is that the response of the material to a constant load is a constant time rate of change of strain. This section considers the form that time derivatives should have to properly describe real materials. Studies in Continuum Mechanics frequently go beyond the notion of a strain rate but the considerations for time differentiation are the same as the ones presented here.

An underlying concept in Continuum Mechanics is that the kinematic behavior of a particle of material is completely controlled by the force, temperature, etc. in the immediate neighborhood of the particle. This concept leads to constitutive equations that apply to material particles. When the time rate of change of configuration is required to describe a particular type of material, the time derivative must apply to a particle, not a fixed spatial location. The remainder of this section presents formulations for finding time derivatives that apply for a particle.

Start by considering a function, f(**X**, **x**, t) representing a time dependent change of configuration where t is time and define the time derivative, $\frac{\mathrm{D}f(\mathbf{X}, \mathbf{x}, t)}{\mathrm{D}t}$, as,

$$\frac{\mathrm{D}f(\mathbf{X}, \mathbf{x}, t)}{\mathrm{D}t} \equiv \frac{d}{dt}\left(f(\mathbf{X}, \mathbf{x}, t)\Big|_{\mathbf{X} = \text{constant}}\right)$$

This derivative is sometimes called the substantial derivative. In the special case where f(**X**, **x**, t) = **x**(**X**,t) this definition yields,

$$\frac{\mathrm{D}\mathbf{x}(\mathbf{X},t)}{\mathrm{D}t} = \frac{\partial \mathbf{x}(\mathbf{X},t)}{\partial t} \equiv \dot{\mathbf{x}}$$

The components of $\dot{\mathbf{x}}$ are the velocity components of the particle identified by the reference coordinate **X**. The definition may now be written as,

$$\frac{\mathrm{D}f(\mathbf{X}, \mathbf{x}, t)}{\mathrm{D}t} \equiv \frac{d}{dt}\left(f(\mathbf{X}, \mathbf{x}, t)\Big|_{\mathbf{X} = \text{constant}}\right) = \frac{\partial}{\partial t}\left(f(\mathbf{X}, \mathbf{x}, t)\right) + \dot{x}_i \cdot \frac{\partial}{\partial x}\left(f(\mathbf{X}, \mathbf{x}, t)\right)$$

This derivative follows the usual chain rule of differentiation so that when.

$$A_\alpha = B_{\alpha\beta} \cdot C_\beta \quad \text{then} \quad \frac{\mathrm{D}A_\alpha}{\mathrm{D}t} = \frac{\mathrm{D}B_{\alpha\beta}}{\mathrm{D}t} \cdot C_\beta + B_{\alpha\beta} \cdot \frac{\mathrm{D}C_\beta}{\mathrm{D}t}$$

and when,

$$a_i = b_{ij} \cdot c_j \quad \text{then} \quad \frac{D a_i}{Dt} = \frac{D b_{ij}}{Dt} \cdot c_j + b_{ij} \cdot \frac{D c_j}{Dt}$$

The chain rule for differentiation can be used to derive useful results as follows. Starting with,

$$\frac{\partial X_\alpha}{\partial x_i} \cdot \frac{\partial x_i}{\partial X_\beta} = \delta_{\alpha\beta}$$

Take the derivative with respect to time to obtain,

$$\frac{D}{Dt}\left(\frac{\partial X_\alpha}{\partial x_i}\right) \cdot \frac{\partial x_i}{\partial X_\beta} = -\frac{\partial X_\alpha}{\partial x_i} \cdot \frac{D}{Dt}\left(\frac{\partial x_i}{\partial X_\beta}\right)$$

Now multiply by $\frac{\partial x_k}{\partial X_\alpha}$ with the implied summation to obtain,

$$\frac{D}{Dt}\left(\frac{\partial x_k}{\partial X_\beta}\right) = -\frac{\partial x_k}{\partial X_\alpha} \cdot \frac{\partial x_i}{\partial X_\beta} \cdot \frac{D}{Dt}\left(\frac{\partial X_\alpha}{\partial x_i}\right)$$

or multiply by $\frac{\partial X_\beta}{\partial x_k}$ with the implied summation to obtain,

$$\frac{D}{Dt}\left(\frac{\partial X_\alpha}{\partial x_k}\right) = -\frac{\partial X_\beta}{\partial x_k} \cdot \frac{\partial X_\alpha}{\partial x_i} \cdot \frac{D}{Dt}\left(\frac{\partial x_i}{\partial X_\beta}\right)$$

Another result that will be useful later is the time derivative of a differential length vector, $dx_i \cdot \vec{v}_i$, fixed to the material. The material velocity at the origin of this vector is $\dot{x}_i \cdot \vec{v}_i$ while the velocity at the tip of the vector is $\left(\dot{x}_i + \frac{\partial \dot{x}_i}{\partial x_j} \cdot dx_j\right) \cdot \vec{v}_i$ so that,

$$\frac{D}{Dt}\left(dx_i \cdot \vec{v}_i\right) = \frac{\partial \dot{x}_i}{\partial x_j} \cdot dx_j \cdot \vec{v}_i$$

or,

$$\frac{D}{Dt}\left(dx_i\right) = \frac{\partial \dot{x}_i}{\partial x_j} \cdot dx_j$$

III.6 STRAIN RATE

Recall the measures of change of distance between two configurations,

$$ds^2 - dS^2 = 2 \cdot e_{ij} \cdot dx_i \cdot dx_j = 2 \cdot E_{\alpha\beta} \cdot dX_\alpha \cdot dX_\beta$$

The case of time dependent motion is considered in this section so the strains are functions of time. If at any time the time rate of change of $ds^2 - dS^2$ vanishes for an arbitrary d**x** then the strain rate components based on the spatial coordinates all vanish. On the other hand, if the time rate of change of $ds^2 - dS^2$ vanishes for arbitrary d**X** then the strain rate components based on the reference coordinates all vanish. Since,

$$dx_i = \frac{\partial x_i}{\partial X_\alpha} \cdot dX_\alpha$$

and $\frac{\partial x_i}{\partial X_\alpha}$ has a bounded inverse the two definitions are physically the same for defining when the strain rate equals zero. However, the two formulations lead to different measures for non-vanishing strain rates. This is demonstrated below.

III.6.1 STRAIN RATE BASED ON REFERENCE COORDINATES

Starting with,

$$ds^2 - dS^2 = 2 \cdot E_{\alpha\beta} \cdot dX_\alpha \cdot dX_\beta$$

A permissible strain rate formulation is found from either of the forms below,

$$\begin{aligned} \frac{D}{Dt}\left(ds^2 - dS^2\right) &= 2 \cdot \frac{\partial E_{\alpha\beta}(\mathbf{X}, t)}{\partial t} \cdot dX_\alpha \cdot dX_\beta \\ &= 2 \cdot \left(\frac{\partial E_{\alpha\beta}(\mathbf{x}, t)}{\partial t} + \frac{\partial E_{\alpha\beta}(\mathbf{x}, t)}{\partial x_i} \cdot \frac{\partial x_i(\mathbf{X}, t)}{\partial \mathbf{t}} \right) \cdot dX_\alpha \cdot dX_\beta \end{aligned}$$

so that strain rate may be determined using,

$$\frac{DE_{\alpha\beta}}{Dt} = \frac{\partial E_{\alpha\beta}(X, t)}{\partial t} = \frac{\partial E_{\alpha\beta}(x, t)}{\partial t} + \frac{\partial E_{\alpha\beta}(x, t)}{\partial x_i} \cdot \frac{\partial x_i(X, t)}{\partial t} = \frac{\partial E_{\alpha\beta}(x, t)}{\partial t} + \frac{\partial E_{\alpha\beta}(x, t)}{\partial x_i} \cdot \dot{x}_i$$

III.6.2 STRAIN RATE BASED ON SPATIAL COORDINATES

Starting with,

$$ds^2 - dS^2 = 2 \cdot e_{ij} \cdot dx_i \cdot dx_j$$

A permissible strain rate definition can be deduced from,

$$\frac{D}{Dt}\left(ds^2 - dS^2\right) = 2 \cdot \frac{D}{Dt}\left(e_{ij} \cdot dx_i \cdot dx_j\right) = 2 \cdot \left(\frac{De_{ij}}{Dt} \cdot dx_i \cdot dx_j + e_{ij} \cdot \frac{D\left(dx_i \cdot dx_j\right)}{Dt}\right)$$

However, this is a rather cumbersome equation to investigate. Fortunately, there is a simpler way to proceed to the desired result. Since dS does not change with time,

$$\frac{D}{Dt}\left(ds^2 - dS^2\right) = \frac{D}{Dt}\left(ds^2\right) = \frac{D}{Dt}\left(dx_i \cdot dx_i\right) = \frac{D}{Dt}\left(\delta_{ij} \cdot dx_i \cdot dx_j\right)$$

The reason for writing this equation with a Kronecker Delta is that the differentiation yields a summation and the contributions from i and j must be kept separate. as the expressions for the strain are coefficients in the summation. This is insured if the indices of the Kronecker delta are kept distinct. Now recalling the result that,

$$\frac{D}{Dt}\left(dx_i\right) = \frac{\partial \dot{x}_i}{\partial x_j} \cdot dx_j$$

the above differentiation becomes,

$$\begin{aligned}
\frac{D}{Dt}\left(\delta_{ij} \cdot dx_i \cdot dx_j\right) &= \delta_{ij} \cdot \left(\frac{\partial \dot{x}_i}{\partial x_k} \cdot dx_k \cdot dx_j + \frac{\partial \dot{x}_j}{\partial x_k} \cdot dx_k \cdot dx_i\right) \\
&= \delta_{kj} \cdot \frac{\partial \dot{x}_k}{\partial x_i} \cdot dx_i \cdot dx_j + \delta_{ik} \cdot \frac{\partial \dot{x}_k}{\partial x_j} \cdot dx_i \cdot dx_j \\
&= \left(\frac{\partial \dot{x}_j}{\partial x_i} + \frac{\partial \dot{x}_i}{\partial x_j}\right) \cdot dx_i \cdot dx_j \equiv 2 \cdot d_{ij} \cdot dx_i \cdot dx_j
\end{aligned}$$

This last equation shows that d_{ij} is an acceptable definition for strain rate. This tensor is called the deformation rate tensor and it determines the instantaneous time rate of change of strain referred to the spatial coordinates. The individual components of d_{ij} are given by,

$$d_{ij} = \frac{1}{2} \cdot \left(\frac{\partial \dot{x}_i}{\partial x_j} + \frac{\partial \dot{x}_j}{\partial x_i}\right)$$

Note that d_{ij} is independent of the reference configuration while $\frac{DE_{\alpha\beta}}{Dt}$ is not. Therefore, unlike the strains, the principal values of dij and $\frac{DE_{\alpha\beta}}{Dt}$ cannot be related one to the other.

III.6.3 ILLUSTRATIVE EXAMPLES FOR STRAIN RATE

Uniaxial Strain:

$$\begin{aligned} x_1 &= K \cdot t \cdot X_1 \\ x_2 &= X_2 \\ x_3 &= X_3 \end{aligned} \qquad \text{or} \qquad \begin{aligned} X_1 &= \frac{x_1}{K \cdot t} \\ X_2 &= x_2 \\ X_3 &= x_3 \end{aligned} \qquad \text{so that} \quad [E_{\alpha\beta}] \equiv \begin{bmatrix} \frac{1}{2} \cdot (K^2 \cdot t^2 - 1) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\left[\frac{DE_{\alpha\beta}}{Dt}\right] \equiv \begin{bmatrix} K^2 \cdot t & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{aligned} \dot{x}_1 &= K \cdot X_1 = \frac{x_1}{t} \\ \dot{x}_2 &= 0 \\ \dot{x}_3 &= 0 \end{aligned} \qquad\qquad [d] = \begin{bmatrix} \frac{1}{t} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Isotropic Strain:

$$\begin{aligned} x_1 &= L \cdot t \cdot X_1 \\ x_2 &= L \cdot t \cdot X_2 \\ x_3 &= L \cdot t \cdot X_3 \end{aligned} \qquad \text{or} \qquad \begin{aligned} X_1 &= \frac{x_1}{L \cdot t} \\ X_2 &= \frac{x_2}{L \cdot t} \\ X_3 &= \frac{x_3}{L \cdot t} \end{aligned} \qquad \text{so that} \quad [E_{\alpha\beta}] \equiv \tfrac{1}{2} \cdot (L^2 \cdot t^2 - 1) \cdot [I]$$

$$\left[\frac{DE_{\alpha\beta}}{Dt}\right] \equiv \begin{bmatrix} L^2 \cdot t & 0 & 0 \\ 0 & L^2 \cdot t & 0 \\ 0 & 0 & L^2 \cdot t \end{bmatrix}$$

$$\dot{x}_1 = L \cdot X_1 = \frac{x_1}{t} \qquad\qquad [d] = \begin{bmatrix} \frac{1}{t} & 0 & 0 \\ 0 & \frac{1}{t} & 0 \\ 0 & 0 & \frac{1}{t} \end{bmatrix}$$
$$\dot{x}_2 = L \cdot X_2 = \frac{x_2}{t}$$
$$\dot{x}_3 = L \cdot X_3 = \frac{x_3}{t}$$

Shear Strain:

$$\begin{aligned} x_1 &= X_1 + M \cdot t \cdot X_2 & \qquad & X_1 = x_1 - M \cdot t \cdot x_2 \\ x_2 &= X_2 & \text{or} \qquad & X_2 = x_2 \\ x_3 &= X_3 & & X_3 = x_3 \end{aligned}$$

so that $$\left[E_{\alpha\beta}\right] \equiv \begin{bmatrix} 0 & \frac{1}{2} \cdot M \cdot t & 0 \\ \frac{1}{2} \cdot M \cdot t & \frac{1}{2} \cdot M^2 \cdot t^2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\left[\frac{DE_{\alpha\beta}}{Dt}\right] \equiv \begin{bmatrix} 0 & \frac{1}{2} \cdot M & 0 \\ \frac{1}{2} \cdot M & M^2 \cdot t & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The principal values are,

$$\left.\frac{DE_{\alpha\beta}}{Dt}\right|_{I} = \tfrac{1}{2} \cdot M \cdot \left(M \cdot t + \sqrt{(M \cdot t)^2 + 1} \right)$$
$$\left.\frac{DE_{\alpha\beta}}{Dt}\right|_{II} = 0$$
$$\left.\frac{DE_{\alpha\beta}}{Dt}\right|_{III} = \tfrac{1}{2} \cdot M \cdot \left(M \cdot t - \sqrt{(M \cdot t)^2 + 1} \right)$$

The principal directions are,

$$\vec{V}_I = \frac{1}{\approx\atop M} \cdot \vec{V}_1 + \frac{M \cdot t + \sqrt{(M \cdot t)^2 + 1}}{\approx\atop M} \cdot \vec{V}_2$$
$$\vec{V}_{II} = \vec{V}_3$$
$$\vec{V}_I = \frac{-M \cdot t - \sqrt{(M \cdot t)^2 + 1}}{\approx\atop M} \cdot \vec{V}_1 + \frac{1}{\approx\atop M} \cdot \vec{V}_2$$

where,

$$\tilde{\tilde{M}} = \sqrt{2 \cdot (M \cdot t)^2 + 2 + 2 \cdot M \cdot t \cdot \sqrt{(M \cdot t)^2 + 1}}$$

$$\dot{x}_1 = M \cdot X_2 = M \cdot x_2 \qquad\qquad [d] = \begin{bmatrix} 0 & \frac{1}{2} \cdot M & 0 \\ \frac{1}{2} \cdot M & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\dot{x}_2 = 0$$

$$\dot{x}_3 = 0$$

The principal values of [d] are,

$$d_I = \tfrac{1}{2} \cdot M$$

$$d_{II} = 0$$

$$d_{III} = -\tfrac{1}{2} \cdot M$$

The principal directions for [d] are,

$$\vec{v}_I = \frac{1}{\sqrt{2}} \cdot (\vec{v}_1 + \vec{v}_2)$$

$$\vec{v}_{II} = 0$$

$$\vec{v}_{III} = \frac{1}{\sqrt{2}} \cdot (-\vec{v}_1 + \vec{v}_2)$$

III.7 COMPATIBILITY CONDITIONS

In preceding developments in this work there has been a tacit assumption that an acceptable motion has been considered. In some analyses of mechanical behavior a solution is initiated by finding a solution for stresses that satisfies force equilibrium (e.g. – the Airy Stress Function) and then the constitutive equations are used to find the strain distribution. Since the strain tensor has nine components and the motion generating these strains is based on three displacement functions, there must be restrictions on the strain distribution in a continuous body. The equations representing these restrictions are referred to as compatibility conditions. As the derivation of these conditions is a purely mathematical matter, only a brief description is presented in this work. A comprehensive presentation of the mathematical formulation is given in P. G. Bergmann's *Introduction to the Theory of Relativity* (Prentice-Hall, 1942).

The base vectors $\vec{V}_\alpha$ and $\vec{v}_i$ are two sets of mutually perpendicular unit length vectors so that,

$$\vec{V}_\alpha \bullet \vec{V}_\beta = \delta_{\alpha\beta}$$
$$\vec{v}_i \bullet \vec{v}_j = \delta_{ij}$$

and,

$$dS^2 = \left(\vec{V}_\alpha \bullet \vec{V}_\beta\right) \cdot dX_\alpha \cdot dX_\beta$$
$$ds^2 = \left(\vec{v}_i \bullet \vec{v}_j\right) \cdot dx_i \cdot dx_j$$

The Chapman distorted base vectors,

$$\vec{\mathbf{V}}_\alpha = \frac{\partial x_i}{\partial X_\alpha} \cdot \vec{v}_i$$

describe the distorted state in terms of the current configuration. That is, $\vec{V}_\alpha \cdot dX_\alpha$ in the reference configuration becomes $\vec{\mathbf{V}}_\alpha \cdot dX_\alpha$ in the current configuration. These base vectors are in general not mutually perpendicular or of unit length so that we define $\mathcal{G}_{\alpha\beta}$ as follows,

$$ds^2 = \left(\vec{\mathbf{V}}_\alpha \bullet \vec{\mathbf{V}}_\beta\right) \cdot dX_\alpha \cdot dX_\beta \equiv \mathbf{G}_{\alpha\beta} \cdot dX_\alpha \cdot dX_\beta$$

The tensor $\mathcal{G}_{\alpha\beta}$ is called the metric for the space and,

$$\mathbf{G}_{\alpha\beta} = \vec{V}_\alpha \bullet \vec{V}_\beta = \frac{\partial x_i}{\partial X_\alpha} \cdot \frac{\partial x_j}{\partial X_\beta} \cdot \vec{v}_i \cdot \vec{v}_j = \frac{\partial x_i}{\partial X_\alpha} \cdot \frac{\partial x_j}{\partial X_\beta} \cdot \delta_{ij} = \frac{\partial x_i}{\partial X_\alpha} \cdot \frac{\partial x_i}{\partial X_\beta}$$

Comparison with,

$$E_{\alpha\beta} = \frac{1}{2} \cdot \left(\frac{\partial x_i}{\partial X_\alpha} \cdot \frac{\partial x_i}{\partial X_\beta} - \delta_{\alpha\beta} \right)$$

shows that,

$$\mathbf{G}_{\alpha\beta} = \delta_{\alpha\beta} + 2 \cdot E_{\alpha\beta}$$

Therefore, the metric for the distorted reference configuration is equal to the undistorted reference configuration metric plus two times the material strain. A similar scheme may be used to find the metric, ϕ_{ij}, for the spatial configuration before the motion occurs.

$$\phi_{ij} \equiv \vec{v}_i \bullet \vec{v}_j = \frac{\partial X_\alpha}{\partial x_i} \cdot \frac{\partial X_\beta}{\partial x_j} \cdot \vec{V}_\alpha \cdot \vec{V}_\beta = \frac{\partial X_\alpha}{\partial x_i} \cdot \frac{\partial X_\beta}{\partial x_j} \cdot \delta_{\alpha\beta} = \frac{\partial X_\alpha}{\partial x_i} \cdot \frac{\partial X_\alpha}{\partial x_j}$$

Comparison with,

$$e_{ij} = \frac{1}{2} \cdot \left(\delta_{ij} - \frac{\partial X_\alpha}{\partial x_i} \cdot \frac{\partial X_\alpha}{\partial x_j} \right)$$

shows that,

$$\phi_{ij} = \delta_{ij} - 2 \cdot e_{ij}$$

The formulations of the metrics ϕ_{ij} and $\mathcal{G}_{\alpha\beta}$ are now used to develop the compatibility conditions. In elementary relativity theory a space is defined by its metric, g_{ij}, which is a function of the coordinates x_i. The relativity derivations are concerned with non-Euclidian spaces while the objective here is to show that ϕ_{ij} and $\mathcal{G}_{\alpha\beta}$ are metrics in Euclidian spaces. Requirements placed on the motion earlier show that the metrics ϕ_{ij} and $\mathcal{G}_{\alpha\beta}$ are positive definite so, from relativity results, the only remaining condition for the spaces to be Euclidian is that the spaces be integrable.

Consider a generic vector that has a fixed direction and length. Move this vector along an arbitrary closed path in the space being investigated. The line integral of the vector along this path must end with the same vector it started with in order for the space to be integratable. The integral whose components must vanish can be expressed in terms of the metric, g_{ij}, and the integral is proportional to the Riemann-Christoffel tensor, R_{smij}, The Riemann-Christoffel tensor may be expressed in terms of the metric, g_{ij}, and its inverse, $\overline{g}_{ij}$, as follows,

$$
\begin{aligned}
R_{smij} = {} & \frac{1}{2}\cdot\left(-\frac{\partial^2 g_{is}}{\partial x_m \partial x_j}+\frac{\partial^2 g_{mi}}{\partial x_j \partial x_s}+\frac{\partial^2 g_{js}}{\partial x_i \partial x_m}-\frac{\partial^2 g_{mj}}{\partial x_i \partial x_s}\right) \\
& +\frac{1}{4}\cdot\left(-\frac{\partial g_{qs}}{\partial x_i}+\frac{\partial g_{is}}{\partial x_q}-\frac{\partial g_{qi}}{\partial x_s}\right)\cdot\overline{g}_{tq}\cdot\left(\frac{\partial g_{mt}}{\partial x_j}+\frac{\partial g_{jt}}{\partial x_m}-\frac{\partial g_{mj}}{\partial x_t}\right) \\
& +\frac{1}{4}\cdot\left(\frac{\partial g_{qs}}{\partial x_j}-\frac{\partial g_{js}}{\partial x_q}+\frac{\partial g_{qj}}{\partial x_s}\right)\cdot\overline{g}_{tq}\cdot\left(\frac{\partial g_{mt}}{\partial x_i}+\frac{\partial g_{it}}{\partial x_m}-\frac{\partial g_{mi}}{\partial x_t}\right)
\end{aligned}
$$

Note that for a Cartesian coordinate system all the components of R_{smij} vanish as the metric, [g], is the identity matrix, [I]. In general, there are only six independent components of R_{smij} and they are indicated in the table below.

COMPONENTS OF THE RIEMANN-CHRISTOFFEL TENSOR

i, j =	**11**	**12**	**13**	**21**	**22**	**23**	**31**	**32**	**33**
ms↓									
11	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**
12	**0**	**A**	**D**	**-A**	**0**	**E**	**-D**	**-E**	**0**
13	**0**	**D**	**B**	**-D**	**0**	**F**	**-B**	**-F**	**0**
21	**0**	**-A**	**-D**	**A**	**0**	**-E**	**D**	**E**	**0**
22	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**
23	**0**	**E**	**F**	**-E**	**0**	**C**	**-F**	**-C**	**0**
31	**0**	**-D**	**-B**	**D**	**0**	**-F**	**B**	**F**	**0**
32	**0**	**-E**	**-F**	**E**	**0**	**-C**	**F**	**C**	**0**
33	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**

Where,

A $= R_{1212}$
B $= R_{1313}$
C $= R_{2323}$
D $= R_{1213}$
E $= R_{1223}$
F $= R_{1323}$

When the metrics ϕ_{ij} and $\mathcal{G}_{\alpha\beta}$ are substituted into R_{smij} , differential equations for $E_{\alpha\beta}$ and e_{ij} are obtained and these are the compatibility conditions.

In the case of strain rate the time rate of change of the metrics ϕ_{ij} and $\mathcal{G}_{\alpha\beta}$ must vanish. For example,

$$\frac{\partial}{\partial t}\left(R_{\alpha\beta\chi\delta}\left(\mathbf{G}_{\kappa\lambda}\left(\mathbf{X}, t\right)\right)\right) = 0$$

and,

$$R_{smij}\left(\delta_{np} + 2 \cdot d_{np}\right) = 0$$

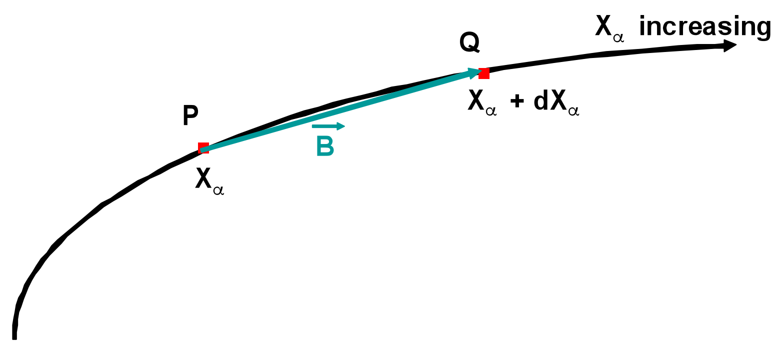
To summarize, the necessary and sufficient conditions that a space be Euclidian are that the six independent components of the Riemann-Christoffel tensor vanish throughout the space and that the metric of the space at every point be positive definite. Obviously, the compatibility conditions can be rather complex differential equations.

IV STRESS

IV.1 PRELIMINARY CONSIDERATIONS

The definition of stress is intimately tied to Newton's linear and angular momentum laws. In this work volume and surface integrals are used so this section presents formulations for differential volume and differential surface elements. These quantities are needed for both the reference and the current configurations as well as their time derivatives.

The sketch below shows a differential length element that is fixed to the material. The element is from a line on which X_α (α = 1, 2 or 3) is changing while the two other coordinates are not changing.



The vector $\vec{B}$ joining points P and Q may be determined in the reference configuration and in the current configuration as,

$$\vec{B}\big|_{\text{REFERENCE}} = \vec{V}_\alpha \cdot dX_\alpha \qquad \text{no } \alpha \text{ summation}$$

$$\vec{B}\big|_{\text{CURRENT}} = \vec{\mathbf{V}}_\alpha \cdot dX_\alpha = \frac{\partial x_i}{\partial X_\alpha} \cdot dX_\alpha \cdot \vec{v}_i \qquad \text{no } \alpha \text{ summation}$$

where $\mathbf{V}_\alpha$ is the Chapman distorted base vector given by,

$$\vec{\mathbf{V}}_\alpha = \frac{\partial x_i}{\partial X_\alpha} \cdot \vec{v}_i$$

The metric for the Chapman distorted base vectors is $\mathcal{G}_{\alpha\beta}$ where,

$$\mathbf{G}_{\alpha\beta} = \vec{\mathbf{V}}_{\alpha} \bullet \vec{\mathbf{V}}_{\beta} = \frac{\partial x_i}{\partial X_\alpha} \cdot \frac{\partial x_i}{\partial X_\beta}$$

and,

$$\vec{B}\Big|_{\text{CURRENT}} = \vec{\mathbf{V}}_{\alpha} \cdot dX_\alpha = \frac{\partial x_i}{\partial X_\alpha} \cdot dX_\alpha \cdot \vec{v}_i = \sqrt{\mathbf{G}_{\alpha\alpha}} \cdot dX_\alpha \qquad \text{no } \alpha \text{ summation}$$

Consequently, for the differential volumes, dVOL, associated with the reference and current coordinate systems are,

$$dVOL\Big|_{\text{REFERENCE}} = dX_1 \cdot dX_2 \cdot dX_3$$

$$dVOL\Big|_{\text{CURRENT}} = \sqrt{\|\mathbf{G}\|} \cdot dX_1 \cdot dX_2 \cdot dX_3$$

Recall from Linear Algebra, that,

$$\|[B]\| = \|[B]^T\|$$
$$\|[A] \cdot [B]\| = \|[A]\| \cdot \|[B]\|$$

to find,

$$dVOL\Big|_{\text{CURRENT}} = \left\|\frac{\partial \mathbf{x}}{\partial \mathbf{X}}\right\| \cdot dX_1 \cdot dX_2 \cdot dX_3 = \left\|\frac{\partial \mathbf{x}}{\partial \mathbf{X}}\right\| \cdot dVOL\Big|_{\text{REFERENCE}}$$

It is clear that the ratio of the current volume to the reference volume from this equation applies to any differential volume.

The time rate of change of the current differential volume may be obtained using the last equation and,

$$\frac{D}{Dt}\left(\frac{\partial x_i(X,t)}{\partial X_\alpha}\right) = \frac{\partial}{\partial t}\left(\frac{\partial x_i(X,t)}{\partial X_\alpha}\right) = \frac{\partial}{\partial X_\alpha}\left(\frac{\partial x_i(X,t)}{\partial t}\right) = \frac{\partial x_j(X.t)}{\partial X_\alpha} \cdot \frac{\partial}{\partial x_j}\left(\frac{\partial x_i(X,t)}{\partial t}\right) = \frac{\partial x_j}{\partial X_\alpha} \cdot \frac{\partial \dot{x}_i}{\partial x_j}$$

The steps are,

$$\frac{D}{Dt}\left(dVOL\big|_{CURRENT}\right) = e_{\alpha\beta\gamma}\cdot\begin{bmatrix}\frac{\partial \dot{x}_1}{\partial x_j}\cdot\frac{\partial x_j}{\partial X_\alpha}\cdot\frac{\partial x_2}{\partial X_\beta}\cdot\frac{\partial x_3}{\partial X_\gamma}\\ +\frac{\partial \dot{x}_2}{\partial x_j}\cdot\frac{\partial x_1}{\partial X_\alpha}\cdot\frac{\partial x_j}{\partial X_\beta}\cdot\frac{\partial x_3}{\partial X_\gamma}\\ +\frac{\partial \dot{x}_3}{\partial x_j}\cdot\frac{\partial x_1}{\partial X_\alpha}\cdot\frac{\partial x_2}{\partial X_\beta}\cdot\frac{\partial x_j}{\partial X_\gamma}\end{bmatrix}\cdot dX_1\cdot dX_2\cdot dX_3$$

Note the summations on the j index can be eliminated by recalling that the determinant of a matrix vanishes when two rows or two columns are identical. Therefore,

$$\frac{D}{Dt}\left(dVOL\big|_{CURRENT}\right) = \frac{\partial \dot{x}_i}{\partial x_i}\cdot\left\|\left[\frac{\partial \mathbf{x}}{\partial \mathbf{X}}\right]\right\| dX_1\cdot dX_2\cdot dX_3 = \frac{\partial \dot{x}_i}{\partial x_i}\cdot dVOL\big|_{CURRENT}$$

since,

$$d_{ii} = \frac{\partial \dot{x}_i}{\partial x_i}$$

there results,

$$\frac{\frac{D}{Dt}\left(dVOL\big|_{CURRENT}\right)}{dVOL\big|_{CURRENT}} = d_{ii}$$

For the determination of expressions for differential elements of surface area take two non-parallel vectors like $\vec{B}$, defined at the beginning of this section, emanating from the same point. Denote these vectors $d\vec{\breve{S}}$ and $d\vec{\hat{S}}$ and then define,

$$d\vec{\breve{S}}\big|_{REFERENCE} = d\breve{S}_\alpha\cdot\vec{V}_\alpha$$

$$d\vec{\hat{S}}\big|_{REFERENCE} = d\hat{S}_\alpha\cdot\vec{V}_\alpha$$

The differential area defined by these vectors is $d\vec{A}$ and it is equal to $d\vec{\breve{S}}\times d\vec{\hat{S}}$ so that,

$$d\vec{A}\big|_{REFERENCE} = e_{\alpha\beta\gamma}\cdot\vec{V}_\alpha\cdot d\breve{S}_\beta\cdot d\hat{S}_\gamma$$

so that the α component is,

$$dA_{\alpha}\Big|_{REFERENCE} = e_{\alpha\beta\gamma} \cdot d\breve{S}_{\beta} \cdot d\widehat{S}_{\gamma}$$

In the current configuration, using the Chapman distorted base vectors, the differential area is derived as follows,

$$d\vec{\breve{S}}\Big|_{CURRENT} = d\breve{S}_{\alpha} \cdot \vec{\mathbf{V}}_{\alpha} = d\breve{S}_{\alpha} \cdot \frac{\partial x_i}{dX_{\alpha}} \cdot \vec{v}_i$$

$$d\vec{\widehat{S}}\Big|_{CURRENT} = d\widehat{S}_{\alpha} \cdot \vec{\mathbf{V}}_{\alpha} = d\widehat{S}_{\alpha} \cdot \frac{\partial x_i}{dX_{\alpha}} \cdot \vec{v}_i$$

and denoting the area component in the current configuration as $da_i\Big|_{CURRENT}$ results in,

$$\begin{aligned}
da_i\Big|_{CURRENT} &= e_{ijk} \cdot \left(d\breve{S}_{\alpha} \cdot \frac{\partial x_j}{\partial X_{\alpha}} \right) \cdot \left(d\widehat{S}_{\beta} \cdot \frac{\partial x_k}{\partial X_{\beta}} \right) \\
&= e_{ljk} \cdot \delta_{il} \cdot \frac{\partial x_j}{\partial X_{\alpha}} \cdot \frac{\partial x_k}{\partial X_{\beta}} \cdot d\breve{S}_{\alpha} \cdot d\widehat{S}_{\beta} \\
&= e_{ljk} \cdot \frac{\partial x_l}{\partial X_{\gamma}} \cdot \frac{\partial X_{\gamma}}{\partial x_1} \cdot \frac{\partial x_j}{\partial X_{\alpha}} \cdot \frac{\partial x_k}{\partial X_{\beta}} \cdot d\breve{S}_{\alpha} \cdot d\widehat{S}_{\beta} \\
&= \left(e_{ljk} \cdot \frac{\partial x_l}{\partial X_{\gamma}} \cdot \frac{\partial x_j}{\partial X_{\alpha}} \cdot \frac{\partial x_k}{\partial X_{\beta}} \right) \cdot \frac{\partial X_{\gamma}}{\partial x_1} \cdot d\breve{S}_{\alpha} \cdot d\widehat{S}_{\beta} \\
&= \left\| \left[\frac{\partial \mathbf{x}}{\partial \mathbf{X}} \right] \right\| \cdot e_{\gamma\alpha\beta} \cdot \frac{\partial X_{\gamma}}{\partial x_1} \cdot d\breve{S}_{\alpha} \cdot d\widehat{S}_{\beta} = \left\| \left[\frac{\partial \mathbf{x}}{\partial \mathbf{X}} \right] \right\| \cdot \frac{\partial X_{\gamma}}{\partial x_1} \cdot dA_{\gamma}\Big|_{REFERENCE}
\end{aligned}$$

The time rate of change of $da_i\Big|_{CURRENT}$ is found using,

$$\frac{D}{Dt}\left(\frac{\partial x_i(X,t)}{\partial X_{\alpha}} \right) = \frac{\partial x_j}{\partial X_{\alpha}} \cdot \frac{\partial \dot{x}_i}{\partial x_j}$$

and the last equation as follows,

$$
\begin{aligned}
\frac{D}{Dt}\left(da_i\big|_{\text{CURRENT}}\right) &= e_{ijk}\cdot\left(\frac{\partial x_j}{\partial X_\alpha}\cdot\frac{D}{Dt}\left(\frac{\partial x_k}{\partial X_\beta}\right)+\frac{D}{Dt}\left(\frac{\partial x_j}{\partial X_\alpha}\right)\cdot\frac{\partial x_k}{\partial X_\beta}\right)\cdot d\breve{S}_\alpha\cdot d\widehat{S}_\beta \\
&= e_{ijk}\cdot\left(\frac{\partial x_j}{\partial X_\alpha}\cdot\frac{\partial x_l}{\partial X_\beta}\cdot\delta_{mk}\cdot\frac{\partial \dot{x}_m}{\partial x_l}+\frac{\partial x_k}{\partial X_\beta}\cdot\frac{\partial x_l}{\partial X_\alpha}\cdot\delta_{mj}\cdot\frac{\partial \dot{x}_m}{\partial x_l}\right)\cdot d\breve{S}_\alpha\cdot d\widehat{S}_\beta \\
&= e_{ijk}\cdot\left(\frac{\partial x_j}{\partial X_\alpha}\cdot\frac{\partial x_l}{\partial X_\beta}\cdot\frac{\partial x_k}{\partial X_\gamma}\cdot\frac{\partial X_\gamma}{\partial x_m}\cdot\frac{\partial \dot{x}_m}{\partial x_l}+\frac{\partial x_k}{\partial X_\beta}\cdot\frac{\partial x_l}{\partial X_\alpha}\cdot\frac{\partial x_j}{\partial X_\gamma}\cdot\frac{\partial X_\gamma}{\partial x_m}\cdot\frac{\partial \dot{x}_m}{\partial x_l}\right)\cdot d\breve{S}_\alpha\cdot d\widehat{S}_\beta \\
&= \left\|\left[\frac{\partial \mathbf{x}}{\partial \mathbf{X}}\right]\right\|\cdot\left(e_{\beta\alpha\gamma}\cdot\frac{\partial X_\gamma}{\partial x_m}\cdot\frac{\partial \dot{x}_m}{\partial x_i}+e_{\alpha\gamma\beta}\cdot\frac{\partial X_\gamma}{\partial x_m}\cdot\frac{\partial \dot{x}_m}{\partial x_l}\right)\cdot d\breve{S}_\alpha\cdot d\widehat{S}_\beta \\
&= -2\cdot\left\|\left[\frac{\partial \mathbf{x}}{\partial \mathbf{X}}\right]\right\|\cdot e_{\gamma\alpha\beta}\cdot\frac{\partial X_\gamma}{\partial x_m}\cdot\frac{\partial \dot{x}_m}{\partial x_i}\cdot d\breve{S}_\alpha\cdot d\widehat{S}_\beta = -2\cdot\frac{\partial \dot{x}_m}{\partial x_i}\cdot da_m\big|_{\text{CURRENT}}
\end{aligned}
$$

Note that the step between the third and fourth lines in the above equation uses the fact that the determinate of a matrix with a repeated row or column vanishes.

IV.2 SOME CONCEPTS FROM MECHANICS

In mechanics of deformable bodies a "physical" particle has special meaning. It is an infinitesimal size volume, dVP, that has a density of ρ (> 0). The particle has a differential mass, dM, equal to $\rho \cdot dVP$ that remains the same when the configuration changes. A body is defined here as a region, R, (includes boundary points) filled (no voids) with a fixed (in time) collection of particles. Each particle *inside* the body has its entire surface, S, in contact with other particles contained in the body. The surface of the body is composed entirely of the boundary particles. Furthermore, the surface of the body contains all the surface elements of all the particles in the body. Each particle is defined by its reference coordinates, X_α and, since the particles have infinitesimal sizes, the body is considered to be a continuum.

Force. $d\vec{\mathbf{F}}$, is a differential vector quantity that, in the absence of other influences, causes a particle to change its velocity vector relative to a spatial coordinate system [**x**] (assumed to be an inertial coordinate system).

The acceleration of a particle, $\vec{\mathbf{A}}$, is the instantaneous time rate of change of the velocity of the particle with respect to a spatial, inertial coordinate system. That is,

$$\vec{\mathbf{A}} = \frac{D}{Dt}\left(\frac{\partial x_i(\mathbf{X},t)}{\partial t}\cdot \vec{v}_i\right) = \frac{\partial^2 x_i(X,t)}{\partial t^2}\cdot \vec{v}_i$$

As $\vec{\mathbf{A}}$ is a vector it may be expressed as,

$$\vec{A} = \ddot{x}_i \cdot \vec{v}_i = \ddot{X}_\alpha \cdot \vec{V}_\alpha$$

where $\ddot{\mathbf{x}}$ and $\ddot{\mathbf{X}}$ are components of acceleration referred to the current and reference coordinate systems, respectively.

With the above definitions Newton's second law for a particle is,

$$d\vec{\mathbf{F}} = dM \cdot \vec{\mathbf{A}}$$

The nature of the force acting on a particle is restricted according to the following definitions.

$$d\vec{\mathbf{F}} = \overset{=}{\vec{\mathbf{F}}} \cdot dM + d\overset{==}{\vec{\mathbf{F}}} + d\overset{===}{\vec{\mathbf{F}}} + \overset{====}{\vec{\mathbf{F}}} \cdot dM$$

where,

$\overset{=}{\vec{\mathbf{F}}} \cdot dM$ is from external sources such as gravity, assumed bounded

$d\vec{\overline{\overline{\mathbf{F}}}}$ is applied to the surface S of the body, assumed continuously distributed

$d\vec{\overline{\overline{\overline{\mathbf{F}}}}}$ is from contact with contiguous particles appearing as opposing equal forces lying on a common line of action

$\vec{\overline{\overline{\overline{\overline{\mathbf{F}}}}}} \cdot dM$ is from particles in R, not from contact appearing as opposing equal forces lying on a common line of action

Now sum all of the forces on particles in R at a fixed time. This summation has no net contribution from $d\vec{\overline{\overline{\overline{F}}}}$ or $\vec{\overline{\overline{\overline{\overline{F}}}}} \cdot dM$ owing to opposing equal forces cancelling. The result is,

$$\int_S d\vec{\overline{\overline{\mathbf{F}}}} + \int_R \vec{\overline{\mathbf{F}}} \cdot dM = \int_R \vec{\mathbf{A}} \cdot dM$$

and with $dM = \rho \cdot dVP$,

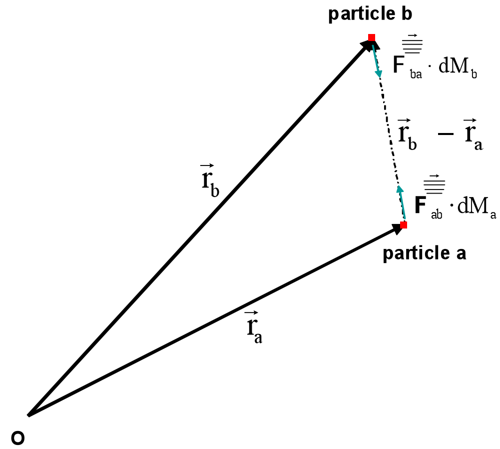
$$\int_S d\vec{\overline{\overline{\mathbf{F}}}} = \int_R \rho \cdot \left(-\vec{\overline{\mathbf{F}}} + \vec{\mathbf{A}} \right) dVP$$

which is Newton's linear momentum law for a body.

The linear momentum law derivation did not take into account the line of action restrictions on forces $d\vec{\overline{\overline{\overline{\mathbf{F}}}}}$ and $\vec{\overline{\overline{\overline{\overline{\mathbf{F}}}}}} \cdot dM$. Satisfaction of these restrictions can be ensured if a second integral is formed. Take an arbitrary point O in the spatial coordinate system and consider a deformed, force-loaded body at a fixed time. Let the vector from point O to the ath particle be $\vec{r}_a$ and form the cross product with Newton's linear momentum law to obtain, with obvious notation,

$$\vec{r}_a \times d\vec{\mathbf{F}}_a = \vec{r}_a \times dM_a \cdot \vec{\mathbf{A}}_a$$

This equation, as it stands, does not give any new information to the theory. When this last equation is summed over all the particles in the body, the contribution from $d\vec{\overline{\overline{\overline{\mathbf{F}}}}}$ vanishes as the contact loads are collinear and opposed. To find the contribution from $\vec{\overline{\overline{\overline{\overline{\mathbf{F}}}}}} \cdot dM$ for the summation consider the sketch below showing two particles, a and b, in the summation.



By postulating,

$$\overset{\vec{\equiv}}{\mathbf{F}}_{ba} \cdot dM_b = -\overset{\vec{\equiv}}{\mathbf{F}}_{ab} \cdot dM_a$$

so that the contribution to the summation is,

$$\vec{r}_b \times \overset{\vec{\equiv}}{\mathbf{F}}_{ba} \cdot dM_b + \vec{r}_a \times \overset{\vec{\equiv}}{\mathbf{F}}_{ab} \cdot dM_b = \left(\vec{r}_b - \vec{r}_a\right) \times \overset{\vec{\equiv}}{\mathbf{F}}_{ba} \cdot dM_b$$

Since the vector $\left(\vec{r}_b - \vec{r}_a\right)$ is parallel to $\overset{\vec{\equiv}}{\mathbf{F}}$ the right hand side of the above equation is zero. This result applies to every pair of particles in the body so that the contribution to the summation from $\overset{\vec{\equiv}}{\mathbf{F}}$ is zero. The equation resulting from the summation is,

$$\int_S \vec{r} \times d\overset{\vec{=}}{\mathbf{F}} = \int_R \rho \cdot \vec{r} \times \left(\overset{\vec{-}}{\mathbf{F}} + \vec{\mathbf{A}}\right) dVP$$

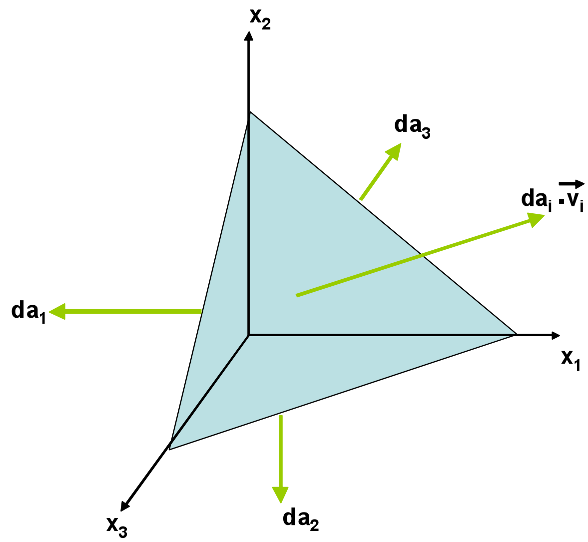
This last equation is Newton's angular momentum law for the body. In the following work the linear and angular momentum laws derived above are assumed valid for every continuous body with constant mass and every sub region therefrom at every instant of time.

IV.3 STRESS

Consider two non-collinear, differential length vectors, $d\vec{x}$ and $d\vec{\vec{x}}$, defined in the current coordinate system. They can be used to define an area $d\vec{a}$ with components da_i that has second order differential length given by,

$$d\vec{a} = d\vec{x} \times d\vec{\vec{x}} = e_{ijk} \cdot d\vec{x}_j \cdot d\vec{\vec{x}}_k \cdot \vec{v}_i = da_i \cdot \vec{v}_i$$

Another way of arriving at this result is through the sketch below showing the inclined vector for the shaded area and its components with implied directions.



In general for a closed single surface, S, $\int_S \vec{n} \cdot dS = 0$ where $\vec{n}$ is the outward, unit normal vector and dS is the differential surface area. Applying this geometric restriction to the four-sided figure in the sketch yields,

$$d\vec{a} = da_i \cdot \vec{v}_i$$

which is the same as the result given at the start of this section.

Now describe the net contact force on the inclined area in its spatial components as $df|_i$. This force has three components so it may be written as,

$$df|_i = s_{ij} \cdot da_j$$

where the coefficients s_{ij} are the stresses and they are, in general, finite but supposed to be bounded. This formulation can be applied to any $d\vec{a}$ so that,

$$d\vec{\overline{\overline{\overline{\mathbf{F}}}}} = df\big|_i \cdot \vec{v}_i = s_{ij} \cdot da_j \cdot \vec{v}_i$$

In this form it is clear that s_{ij} is a second rank tensor.

For subsequent work two additional requirements are imposed on $d\vec{\overline{\overline{\overline{\mathbf{F}}}}}$. They are,

1. $d\vec{\overline{\overline{\overline{\mathbf{F}}}}}$ is distributed within the region R so that s_{ij} and its first derivative are continuous.
2. As an element of area, parallel to the tangent plane at any point on the surface of the region R, is taken at interior points closer and closer to the point P, $d\vec{\overline{\overline{\overline{\mathbf{F}}}}}$ approaches $d\vec{\overline{\overline{\overline{\mathbf{F}}}}}$ for the surface point in the limit.

IV.4 RESTRICTIONS ON THE STRESS TENSOR FROM NEWTON'S LAWS

Any stress distribution in a body or sub body must satisfy Newton's linear and angular momentum laws. The ways that these laws restrict stress distributions is presented in this section. Recall that stress, s_{ij}, is defined from,

$$d\overset{\Rightarrow}{\mathbf{F}} = s_{ij} \cdot da_j \cdot \vec{v}_i$$

and it may be used with other continuity conditions already imposed to introduce stress into the linear and angular momentum laws. The linear momentum law,

$$\int_S d\overset{\Rightarrow}{\mathbf{F}} = \int_R \rho \cdot \left(-\overset{\rightharpoonup}{\mathbf{F}} + \vec{\mathbf{A}}\right) dVP$$

becomes,

$$\int_S s_{ij} \cdot \vec{v}_i \cdot da_j = \int_R \rho \cdot \left(-\bar{f}_i + \ddot{x}_i\right) \cdot \vec{v}_i \cdot dVP$$

Gauss' theorem can be applied to the left hand side of this equation to obtain,

$$\int_S s_{ij} \cdot \vec{v}_i \cdot da_j = \int_R \frac{\partial}{\partial x_j}\left(s_{ij} \cdot \vec{v}_i\right) \cdot dVP$$

so that,

$$\int_R \left(-\rho \cdot \bar{f}_i + \rho \cdot \ddot{x}_i - \frac{\partial s_{ij}}{\partial x_j}\right) \cdot \vec{v}_i \cdot dVP = 0$$

This equation must apply for every sub region of R so the integrand must vanish everywhere in R. This yields the usual form for the linear momentum law,

$$\frac{\partial s_{ij}}{\partial x_j} + \rho \cdot \bar{f}_i = \rho \cdot \ddot{x}_i$$

and it must be satisfied everywhere in R.

The angular momentum law gives a second restriction on the stress tensor. Following the same procedure as for the linear momentum law yields,

$$\int_S \vec{r} \times s_{ij} \cdot \vec{v}_i \cdot da_j = \int_R \rho \cdot \vec{r} \times \left(-\bar{f}_i + \ddot{x}_i \right) \cdot \vec{v}_i \cdot dVP$$

After applying Gauss' theorem to the left hand side of this equation and converting the vectors to component form, there results,

$$\int_R \left(e_{ijk} \cdot \frac{\partial \left(r_j \cdot s_{kq} \right)}{\partial x_q} - \rho \cdot e_{ijk} \cdot r_j \cdot \left(-\bar{f}_k + \ddot{x}_k \right) \right) \cdot \vec{v}_i \cdot dVP = 0$$

Since this is valid for every sub region of R the integrand must vanish in R so that,

$$e_{ijk} \cdot \left(s_{kq} \cdot \frac{\partial r_j}{\partial x_q} + r_j \cdot \left(\frac{\partial s_{kq}}{\partial x_q} + \rho \cdot \bar{f}_k - \rho \cdot \ddot{x}_k \right) \right) = 0$$

The term multiplying r_j vanishes in view of the linear momentum law and,

$$\frac{\partial r_j}{\partial x_q} = \delta_{jq}$$

so the angular momentum law reduces to,

$$e_{ijk} \cdot s_{kj} = 0$$

or,

$$s_{kj} = s_{jk}$$

and the restriction from the angular momentum law on the stress tensor is that it is symmetric.

IV.5 STRESS TENSORS IN THE REFERENCE CONFIGURATION

From a physical viewpoint it is natural to develop the concept of stress from the differential current contact force acting on a differential area in the current configuration. That procedure was followed in the preceding section and the stress, s_{ij}, was defined. In this section a stress is defined using the current differential contact force acting on a differential area in the reference configuration. This differential area is the area in the reference configuration that corresponds to the differential area in the current configuration that the differential contact force is acting upon. To accomplish this some groundwork must precede the definition of the new stress.

The differential contact force, $d\vec{\overline{\overline{\mathbf{F}}}}$, may be resolved in the reference configuration as,

$$d\vec{\overline{\overline{\mathbf{F}}}} = dF_{\alpha} \cdot \vec{V}_{\alpha}$$

The differential area vector $d\vec{A}\big|_{REF}$ of an element in the reference configuration can also be resolved into its reference configuration components as,

$$d\vec{A}\big|_{REF} = dA_{\alpha}\big|_{REF} \cdot \vec{V}_{\alpha}$$

Now a material stress $\mathbf{S}_{\alpha\beta}$ is defined using,

$$d\vec{\overline{\overline{\overline{F}}}}_{\alpha} = \mathbf{S}_{\alpha\beta} \cdot dA_{\beta}\big|_{REF}$$

and $\mathbb{S}_{\alpha\beta}$ as well as $\dfrac{\partial \mathbf{S}_{\alpha\beta}}{\partial X_{\gamma}}$ are assumed to be continuous. Also, $d\vec{\overline{\overline{\overline{F}}}} \to d\vec{\overline{\overline{F}}}$ as the surface S is approached so $d\vec{\overline{\overline{\overline{F}}}} = \mathbf{S}_{\alpha\beta} \cdot dA_{\beta}\big|_{REF} \cdot \vec{V}_{\alpha}$. Since mass is conserved,

$$dM = \rho \cdot dVR \equiv \rho_O \cdot dVR\big|_{REF}$$

where r_O is the mass density of the particle in the reference configuration and $R \to R_O$ while $S \to S_O$ so the linear momentum law becomes,

$$\int_{S_O} \mathbf{S}_{\alpha\beta} \cdot \vec{V}_{\alpha} \cdot dA_{\beta}\big|_{REF} + \int_{R_O} \rho_O \cdot \left(\vec{\overline{\overline{\overline{F}}}}_{\alpha} - \ddot{X}\right) \cdot \vec{V}_{\alpha} \cdot dVR\big|_{REF} = 0$$

This form results after the same arguments used in the spatial case concerning the forces on a particle are introduced. Gauss' theorem may be applied to the first term to obtain,

$$\int_{S_O} \mathbf{S}_{\alpha\beta} \cdot \vec{V}_\alpha \cdot dA_\beta \Big|_{REF} = \int_{R_O} \frac{\partial}{\partial X_\beta}\left(\mathbf{S}_{\alpha\beta} \cdot \vec{V}_\alpha\right) dVR\Big|_{REF} = \int_{R_O} \vec{V}_\alpha \cdot \frac{\partial}{\partial X_\beta}\left(\mathbf{S}_{\alpha\beta}\right) \cdot dVR\Big|_{REF}$$

so that,

$$\int_{R_O} \vec{V}_\alpha \cdot \left(\frac{\partial}{\partial X_\beta}\left(\mathbf{S}_{\alpha\beta}\right) + \rho_O \cdot \left(\overline{\overline{\overline{F}}}_\alpha - \ddot{X} \right) \right) \cdot dVR\Big|_{REF} = 0$$

Since this integral must vanish for every sub region of R_O, the integrand must vanish at every point in R_O so,

$$\frac{\partial}{\partial X_\beta}\left(\mathbf{S}_{\alpha\beta}\right) + \rho_O \cdot \overline{\overline{\overline{F}}}_\alpha = \rho_O \cdot \ddot{X}$$

The linear momentum law requires that this equation be valid in R_O.

To determine restrictions on $\mathbb{S}_{\alpha\beta}$ imposed by the angular momentum law, the cross product of the position vector $\vec{R}$ with the linear momentum law is formed and the arguments concerning lines of action between particles used in the spatial formulation.are introduced. Note that $\vec{R}$ must be measured in the spatial coordinate system in order to be consistent with Newton's law. The result is,

$$\int_{S_O} e_{\alpha\beta\gamma} \cdot R_\beta \cdot \mathbf{S}_{\gamma\delta} \cdot \vec{V}_\alpha \cdot dA_\delta \Big|_{REF} + \int_{R_O} e_{\alpha\beta\gamma} \cdot R_\beta \cdot \rho_O \cdot \left(\overline{\overline{\overline{F}}}_\alpha - \ddot{X} \right) \cdot \vec{V}_\alpha \cdot dVR\Big|_{REF} = 0$$

Application of Gauss' theorem yields,

$$\int_{R_O} e_{\alpha\beta\gamma} \cdot \left(\frac{\partial}{\partial X_\delta}\left(R_\beta \cdot \mathbf{S}_{\gamma\delta}\right) + R_\beta \cdot \rho_O \cdot \left(\overline{\overline{\overline{F}}}_\alpha - \ddot{X} \right) \right) \cdot \vec{V}_\alpha \cdot dVR\Big|_{REF} = 0$$

so that,

$$\int_{R_O} e_{\alpha\beta\gamma} \cdot \left(\frac{\partial R_\beta}{\partial X_\delta} \cdot \mathbf{S}_{\gamma\delta} + R_\beta \cdot \left(\frac{\partial \mathbf{S}_{\gamma\delta}}{\partial X_\delta} + \rho_O \cdot \left(\overline{\overline{\overline{F}}}_\alpha - \ddot{X} \right) \right) \right) \cdot \vec{V}_\alpha \cdot dVR\Big|_{REF} = 0$$

Since this equation must be satisfied for every sub region of R_O,

$$e_{\alpha\beta\gamma} \cdot \frac{\partial R_\beta}{\partial X_\delta} = 0 \qquad \text{for } \alpha = 1, 2, 3$$

This equation gives the restriction on $\mathbb{S}_{\alpha\beta}$ from the angular momentum law. Recalling that,

$$\frac{\partial R_\alpha}{\partial X_\delta} = \frac{\partial x_i}{\partial X_\delta} \cdot \vec{v}_i \bullet \vec{V}_\alpha$$

it is clear that $\mathbb{S}_{\alpha\beta}$ is *not* generally symmetric.

To obtain a relationship between $\mathbb{S}_{\alpha\beta}$ and s_{ij} the following results presented earlier are used as a starting point.

$$d\overline{\overline{\mathbf{F}}} = s_{ij} \cdot da_j\big|_{CURRENT} \cdot \vec{v}_i = \mathbb{S}_{\alpha\beta} \cdot dA_\beta\big|_{REF} \cdot \vec{V}_\alpha$$

The quantity $da_j\big|_{CURRENT}$ is a differential area in the current configuration while $dA_\beta\big|_{REF}$ is the corresponding differential area in the reference configuration. The relation between these areas has been derived in an earlier section and is repeated here,

$$da_i\big|_{CURRENT} = \left\| \left[\frac{\partial \mathbf{x}}{\partial \mathbf{X}} \right] \right\| \cdot \frac{\partial X_\gamma}{\partial x_1} \cdot dA_\gamma\big|_{REF}$$

When the last two equations are combined, the result is,

$$\mathbb{S}_{\alpha\beta} = \vec{v}_i \bullet \vec{V}_\alpha \cdot \left\| \left[\frac{\partial \mathbf{x}}{\partial \mathbf{X}} \right] \right\| \cdot \frac{\partial X_\beta}{\partial x_j} \cdot s_{ij}$$

The complexity of using $\mathbb{S}_{\alpha\beta}$ in analysis, primarily because it is not symmetric, makes this definition for stress virtually unused.

In order to be able to formulate theories expressed in the reference configurations coordinates, the Kirchhoff stress tensor, $S_{\alpha\beta}$, often appears in the literature. It has the definition,

$$d\overline{\overline{\mathbf{F}}} = \mathbf{S}_{\alpha\beta} \cdot \vec{\mathbf{V}}_\alpha \cdot dA_\beta\big|_{REF} = \mathbf{S}_{\alpha\beta} \cdot \frac{\partial x_i}{\partial X_\alpha} \cdot \vec{v}_i \cdot dA_\beta\big|_{REF}$$

and

$$d\overline{\overline{\mathbf{F}}} = s_{ij} \cdot da_j\big|_{CURRENT} \cdot \vec{v}_i$$

Equating the right hand sides of the last two equations and using,

$$da_i\Big|_{CURRENT} = \left\| \left[\frac{\partial \mathbf{x}}{\partial \mathbf{X}} \right] \right\| \cdot \frac{\partial X_\gamma}{\partial x_1} \cdot dA_\gamma \Big|_{REF}$$

yields,

$$s_{ij} \cdot \left\| \left[\frac{\partial \mathbf{x}}{\partial \mathbf{X}} \right] \right\| \cdot \frac{\partial X_\beta}{\partial x_1} \cdot dA_\beta \Big|_{REF} = \mathsf{S}_{\alpha\beta} \cdot \frac{\partial x_i}{\partial X_\alpha} \cdot dA_\beta \Big|_{REF}$$

now multiplying by $\frac{\partial x_k}{\partial X_\beta}$ with implied summation to get,

$$s_{ik} = \left\| \left[\frac{\partial \mathbf{X}}{\partial \mathbf{x}} \right] \right\| \cdot \mathsf{S}_{\alpha\beta} \cdot \frac{\partial x_i}{\partial X_\alpha} \cdot \frac{\partial x_k}{\partial X_\beta} \cdot dA_\beta \Big|_{REF}$$

This equation may be inverted to obtain,

$$\mathsf{S}_{\alpha\beta} = s_{ij} \cdot \left\| \left[\frac{\partial \mathbf{x}}{\partial \mathbf{X}} \right] \right\| \cdot \frac{\partial X_\alpha}{\partial x_i} \cdot \frac{\partial X_\beta}{\partial x_j}$$

As s_{ij} is symmetric, so is $\mathsf{S}_{\alpha\beta}$.

The linear momentum law for $\mathsf{S}_{\alpha\beta}$ may be obtained by substituting the second-to-last equation into,

$$\frac{\partial s_{ij}}{\partial x_j} + \rho \cdot \bar{f}_i = \rho \cdot \ddot{x}_i$$

The angular momentum law is satisfied owing to the requirement of symmetry of s_{ij} and thus of $\mathsf{S}_{\alpha\beta}$.

IV.6 EXTREME VALUES OF THE SHEAR AND NORMAL STRESSES

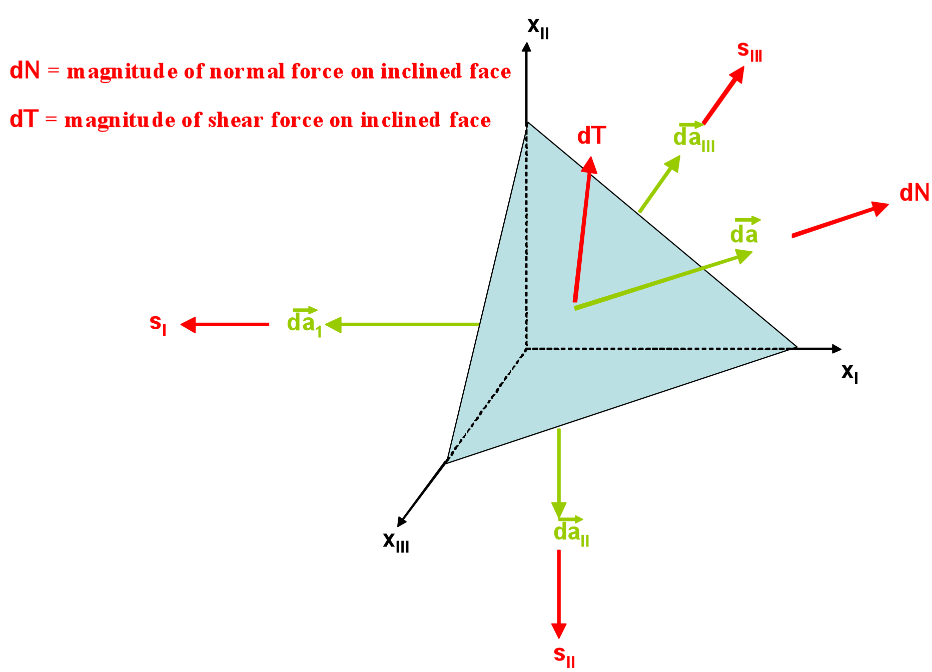
This section determines several useful results concerning stresses. The results are applicable to any real, symmetric second rank matrix. To simplify the presentation the derivations use the principal stresses s_I, s_{II}, s_{III} which can always be found. Assume the principle stresses are ordered so that,

$$s_I \geq s_{II} \geq s_{III}$$

and the stress matrix is,

$$[s] \equiv \begin{bmatrix} s_I & 0 & 0 \\ 0 & s_{II} & 0 \\ 0 & 0 & s_{III} \end{bmatrix}$$

Now consider a four sided, differential size element with three of the sides perpendicular to the three, perpendicular principal directions. The element is sketched below.



In the figure above the stresses on the three principal stress faces are s_I, s_{II}, s_{III}. The areas of the three faces are $d\vec{a}_I, d\vec{a}_{II}, d\vec{a}_{III}$ and the area of the inclined face is,

$$d\vec{a} = -\left(d\vec{a}_I + d\vec{a}_{II} + d\vec{a}_{III}\right)$$

The forces on the three faces are $s_I \cdot d\vec{a}_I, s_{II} \cdot d\vec{a}_{II}, s_{III} \cdot d\vec{a}_{III}$. The force on the inclined area is resolved into a normal component of magnitude, $d\mathbf{N}$, and a shear component parallel to the plane of the inclined face of magnitude, $d\mathbf{T}$. Now define,

$$N = \text{normal stress} = \frac{d\mathbf{N}}{|d\vec{a}|}$$

$$T = \text{shear stress} = \frac{d\mathbf{T}}{|d\vec{a}|}$$

In order to resolve the forces into their components, the following definitions are introduced,

$$\nu_I = \frac{|d\vec{a}_I|^2}{|d\vec{a}_I|^2 + |d\vec{a}_{II}|^2 + |d\vec{a}_{III}|^2} = \cos^2\left(\angle\, d\vec{a}, d\vec{a}_I\right)$$

$$\nu_{II} = \frac{|d\vec{a}_{II}|^2}{|d\vec{a}_I|^2 + |d\vec{a}_{II}|^2 + |d\vec{a}_{III}|^2} = \cos^2\left(\angle\, d\vec{a}, d\vec{a}_{II}\right)$$

$$\nu_{III} = \frac{|d\vec{a}_{III}|^2}{|d\vec{a}_I|^2 + |d\vec{a}_{II}|^2 + |d\vec{a}_{III}|^2} = \cos^2\left(\angle\, d\vec{a}, d\vec{a}_{III}\right)$$

The requirement that the net force on the four-sided element be zero is,

$$s_I^{\,2} \cdot \nu_I + s_{II}^{\,2} \cdot \nu_{II} + s_{III}^{\,2} \cdot \nu_{III} = N^2 + T^2$$

Force equilibrium in the direction perpendicular to the inclined face gives,

$$s_I \cdot \nu_I + s_{II} \cdot \nu_{II} + s_{III} \cdot \nu_{III} = N$$

From the definitions of ν_I, ν_{II}, ν_{III} there results,

$$\nu_I + \nu_{II} + \nu_{III} = 1$$

These last three equations may be written in matrix form as,

$$\begin{bmatrix} s_I^{\,2} & s_{II}^{\,2} & s_{III}^{\,2} \\ s_I & s_{II} & s_{III} \\ 1 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} \nu_I \\ \nu_{II} \\ \nu_{III} \end{bmatrix} = \begin{bmatrix} N^2 + T^2 \\ N \\ 1 \end{bmatrix}$$

The determinant of the coefficient matrix is,

$$\left\|\begin{bmatrix} s_I^{\,2} & s_{II}^{\,2} & s_{III}^{\,2} \\ s_I & s_{II} & s_{III} \\ 1 & 1 & 1 \end{bmatrix}\right\| = (s_I - s_{II})\cdot(s_{II} - s_{III})\cdot(s_I - s_{III})$$

First consider the case of the principal values of stress having distinct values so that $s_I > s_{II} > s_{III}$ so that the determinant is greater than zero. In this case the solution for ν_I, ν_{II}, ν_{III} is,

$$\nu_I = \frac{T^2 + (N - s_{II})\cdot(N - s_{III})}{(s_I - s_{II})\cdot(s_I - s_{III})}$$

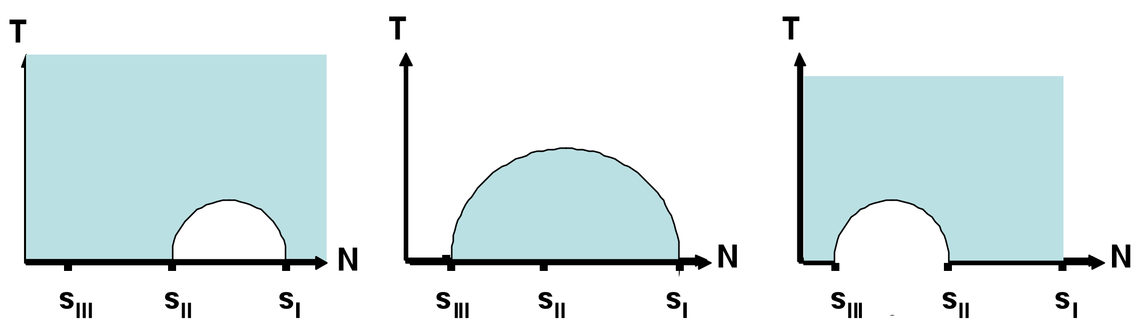
$$\nu_{II} = -\frac{T^2 + (N - s_{III})\cdot(N - s_I)}{(s_{II} - s_{III})\cdot(s_I - s_{II})}$$

$$\nu_{III} = \frac{T^2 + (N - s_I)\cdot(N - s_{II})}{(s_I - s_{III})\cdot(s_{II} - s_{III})}$$

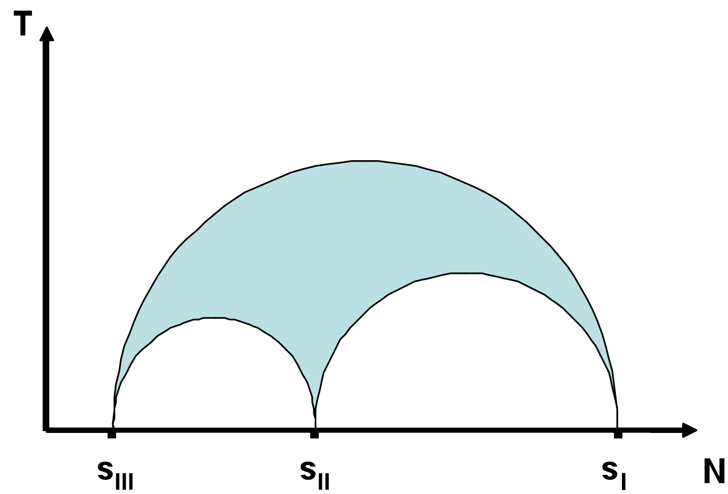
The values of ν_I, ν_{II}, ν_{III} must all be greater than zero in order that the angles they are defined by be real. Therefore,

$$T^2 + (N - s_{II})\cdot(N - s_{III}) \geq 0$$
$$T^2 + (N - s_{III})\cdot(N - s_I) \leq 0$$
$$T^2 + (N - s_I)\cdot(N - s_{II}) \geq 0$$

When T is plotted versus N, the three inequalities determine regions that are admissible. The shaded regions for each inequality are shown in the sketches below.



When the restrictions from the three inequalities are combined, the result is shown below.



The limits of the admissible values for the normal stresses are seen to be $s_I < N < s_{III}$ while the limits of the shear stress are $0 < T < \frac{1}{2} \cdot (s_I - s_{III})$, These useful results are the bases of criteria used in machine design calculations to avoid failures of structures.

The derivation for cases of repeated principal stresses follows the same scheme. The results are consistent with a graphical interpretation of the above sketch. That is, when two principal stresses are equal and distinct from the third, the sketch reduces to a circle. If $s_I = s_{II} = s_{III}$ then the admissible region degenerates to a point.

V STRESS RATE

V.1 PRELIMINARY CONSIDERATIONS

The concept of stress rate pursued here is for use in constitutive equations where the response of a material depends on its loading as well as the time rate of change of loading. Some viscoelastic materials, by definition, have physical behaviors that require consideration of the current stress state and the time rate of change of stress. The definition of an acceptable stress rate, for example, should not be influenced by a rigid body motion if the stresses are not altered relative to the body.

Other physical considerations enter into the development of a mathematical definition of stress rate. To illustrate, imagine that a differential area element is composed of a layer with a fixed number of particles. During a general motion this layer will change size. In the simple case where the stress field is a constant, uniform pressure the normal stress on the layer does not change and the stress rate could be considered to be zero. On the other hand, the load per particle changes and, assuming the material response is influenced by the load per particle, the stress rate will be nontrivial. It is not surprising that different definitions for stress rate appear in the literature.

Results from Section IV suggest that it is more satisfying physically to restrict considerations of stress to the definition of the current configuration stress, s_{ij}, given by,

$$d\vec{\underline{\underline{\mathbf{F}}}} = s_{ij} \cdot \vec{v}_i \cdot da_j$$

In this section the definitions for stress rate are all given in terms of s_{ij}.

Since any acceptable definition of stress rate must be insensitive to certain cases of rigid body motion, it is helpful to introduce vorticity at this point in the development. Vorticity is a tensor related to the rotation rate of a body. Start with the earlier result that,

$$d_{ij} = \tfrac{1}{2} \cdot \left(\frac{\partial \dot{x}_i}{\partial x_j} + \frac{\partial \dot{x}_j}{\partial x_i} \right)$$

and define the vorticity tensor, ω_{ij}, using,

$$\frac{\partial \dot{x}_i}{\partial x_j} = d_{ij} - \omega_{ij}$$

so that,

$$\omega_{ij} = -\tfrac{1}{2} \cdot \left(\frac{\partial \dot{x}_i}{\partial x_j} - \frac{\partial \dot{x}_j}{\partial x_i} \right)$$

This equation shows that the vorticity matrix, $[\omega]$, is asymmetric with the following properties,

$$\omega_{ij} = -\omega_{ji} \qquad \text{if } i \neq j$$
$$\omega_{ij} = 0 \qquad \text{if } i = j$$

Consequently, $[\omega]$ has the form,

$$[\omega] = \begin{bmatrix} 0 & \omega_{12} & \omega_{13} \\ -\omega_{12} & 0 & \omega_{23} \\ -\omega_{13} & -\omega_{23} & 0 \end{bmatrix}$$

with,

$$\omega_{12} = -\tfrac{1}{2} \cdot \left(\frac{\partial \dot{x}_1}{\partial x_2} - \frac{\partial \dot{x}_2}{\partial x_1} \right)$$
$$\omega_{31} = -\tfrac{1}{2} \cdot \left(\frac{\partial \dot{x}_3}{\partial x_1} - \frac{\partial \dot{x}_1}{\partial x_3} \right)$$
$$\omega_{23} = -\tfrac{1}{2} \cdot \left(\frac{\partial \dot{x}_2}{\partial x_3} - \frac{\partial \dot{x}_3}{\partial x_2} \right)$$

In many developments of fluid mechanics theory the vorticity vector is defined in conventional vector notation as the curl of the velocity vector,

$$\begin{aligned} \nabla \times \vec{\dot{x}} &= \left(\vec{i} \cdot \frac{\partial}{\partial x_1} + \vec{j} \cdot \frac{\partial}{\partial x_2} + \vec{k} \cdot \frac{\partial}{\partial x_3} \right) \times \left(\vec{i} \cdot \dot{x}_1 + \vec{j} \cdot \dot{x}_2 + \vec{k} \cdot \dot{x}_3 \right) \\ &= \vec{i} \cdot \left(\frac{\partial \dot{x}_3}{\partial x_2} - \frac{\partial \dot{x}_2}{\partial x_3} \right) + \vec{j} \cdot \left(\frac{\partial \dot{x}_1}{\partial x_3} - \frac{\partial \dot{x}_3}{\partial x_1} \right) + \vec{k} \cdot \left(\frac{\partial \dot{x}_2}{\partial x_1} - \frac{\partial \dot{x}_1}{\partial x_2} \right) \\ &= \vec{i} \cdot 2 \cdot \omega_{23} + \vec{j} \cdot 2 \cdot \omega_{31} + \vec{k} \cdot 2 \cdot \omega_{12} \equiv \vec{\omega} \end{aligned}$$

leading to,

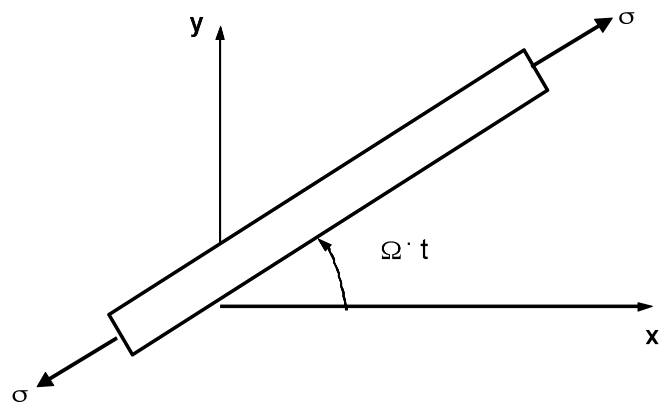
$$\omega_1 = 2 \cdot \omega_{23}$$
$$\omega_2 = 2 \cdot \omega_{31}$$
$$\omega_3 = 2 \cdot \omega_{12}$$

or,

$$\omega_i = e_{ijk} \cdot \omega_{jk}$$

The usual fluid mechanics formulation is consistent with the presentation in this work.

A simple example will serve to illustrate the interaction of the motion and the rate of change of stress. Consider the sketch below of an elastic bar with a uniform, constant, axial stress, σ. The axis of the bar is initially (t = 0) in the x direction and is being rotated in the xy plane at the constant rate of rotation, Ω.



It is clear that for this example both the deformation rate, [d], and the stress rate vanish. The substantial time derivative of the stress is,

$$\frac{D s_{ij}}{Dt} = \frac{\partial s_{ij}}{\partial t} + \frac{\partial s_{ij}}{\partial x_k} \cdot \dot{x}_k = \frac{\partial}{\partial t}\begin{bmatrix} \frac{1}{2}\cdot\sigma\cdot(1+\cos(2\cdot\Omega\cdot t)) & \frac{1}{2}\cdot\sigma\cdot\sin(2\cdot\Omega\cdot t) & 0 \\ \frac{1}{2}\cdot\sigma\cdot\sin(2\cdot\Omega\cdot t) & \frac{1}{2}\cdot\sigma\cdot(1-\cos(2\cdot\Omega\cdot t)) & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -\sigma\cdot\Omega\cdot\sin(\Omega\cdot t) & \sigma\cdot\Omega\cdot\cos(\Omega\cdot t) & 0 \\ \sigma\cdot\Omega\cdot\cos(\Omega\cdot t) & \sigma\cdot\Omega\cdot\sin(\Omega\cdot t) & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Although this is the correct answer for the way the spatial stress components are changing, it does not represent a valid measure for the stress rate to be included in a constitutive equation.

For example, suppose a constitutive equation is postulated to have the stress rate equal to a function of the deformation rate. If the motion considered is a rigid body motion then all the components of [d] vanish but as shown above the components of $\frac{D s_{ij}}{Dt}$ do not vanish. Another way of looking at this is that we have defined strain and deformation rate to vanish when the motion is a rigid body motion but the same is not true of $\frac{D s_{ij}}{Dt}$. Consequently, a different formulation for an acceptable stress rate is required. To help in the development of acceptable forms for the stress rate, a few preliminary results are presented below.

The motion of a differential length line element fixed in the material is now investigated. Consider two particles P and Q. In the current configuration P has coordinates x_i and Q has coordinates $x_i + dx_i$ and the vector joining the particles, $\vec{A}$, is,

$$\vec{A} = dx_i \cdot \vec{v}_i \quad \text{and} \quad \left|\vec{A}\right| = dx_i \cdot dx_i$$

Since $\vec{v}_i$ is independent of time,

$$\frac{D\vec{A}}{Dt} = \frac{\partial \dot{x}_i}{\partial x_j} \cdot \vec{v}_i \cdot dx_j$$

Define a second differential length vector, $\vec{B}$, emanating from x_i but with a different change of spatial coordinates, $d\overline{x}_i$, to particle M so that,

$$\vec{B} = d\overline{x}_i \cdot \vec{v}_i \quad , \quad \left|\vec{B}\right| = d\overline{x}_i \cdot d\overline{x}_i \quad \text{and} \quad \frac{D\vec{B}}{Dt} = \frac{\partial \dot{x}_i}{\partial x_j} \cdot \vec{v}_i \cdot d\overline{x}_j$$

The dot product of these two vectors yields,

$$\vec{A} \bullet \vec{B} = dx_i \cdot d\overline{x}_i$$

and,

$$\begin{aligned}\frac{D}{Dt}\left(\vec{A} \bullet \vec{B}\right) &= \frac{\partial \dot{x}_i}{\partial x_k} \cdot dx_k \cdot d\overline{x}_i + \frac{\partial \dot{x}_i}{\partial x_k} \cdot dx_i \cdot d\overline{x}_k = \frac{\partial \dot{x}_i}{\partial x_k} \cdot \left(dx_k \cdot d\overline{x}_i + dx_i \cdot d\overline{x}_k\right) \\ &= \left(d_{ik} - \omega_{ik}\right) \cdot \left(dx_k \cdot d\overline{x}_i + dx_i \cdot d\overline{x}_k\right)\end{aligned}$$

Now note that,

$$\omega_{ik} \cdot \left(dx_k \cdot d\overline{x}_i + dx_i \cdot d\overline{x}_k\right) = \left(\omega_{ik} + \omega_{ki}\right) \cdot dx_k \cdot d\overline{x}_i = 0$$

owing to the asymmetry of ω_{ij}. Combining the last two equations gives,

$$\frac{D}{Dt}\left(\vec{A} \bullet \vec{B}\right) = d_{ik} \cdot \left(dx_k \cdot d\overline{x}_i + dx_i \cdot d\overline{x}_k\right) = 2 \cdot d_{ik} \cdot dx_i \cdot d\overline{x}_k$$

owing to the symmetry of d_{ij}.

For the special case at particle P where the x_i directions are in the principal directions for [d], this makes [d] a diagonal matrix. Let $\vec{A}$ be directed in one of the principal directions while $\vec{B}$ is directed in one of the other principal directions. The last equation shows that, in this specific case, $\frac{D}{Dt}\left(\vec{A} \bullet \vec{B}\right) = 0$ so that the vectors are not only perpendicular but their rate of change with time is such that they remain perpendicular through first order terms in Δt. This result shows that a cube aligned with the principal directions of [d] at time t remains, through first order terms in Δt, a rectangular parallelepiped at $t + \Delta t$.

The next item for investigation is to find the time derivative of a unit length vector fixed in the material and directed in a principal direction of [d]. Take the vector $\vec{A}$ defined above and form a unit length vector $\vec{\tilde{A}}$ parallel to $\vec{A}$ as,

$$\vec{\tilde{A}} = \frac{\vec{A}}{\left|\vec{A}\right|} = \frac{\vec{v}_i \cdot dx_i}{\sqrt{dx_m \cdot dx_m}}$$

The time derivative of this vector is,

$$\frac{D\vec{\tilde{A}}}{Dt} = \vec{v}_i \cdot \left(\frac{\frac{\partial \dot{x}_i}{\partial x_j} \cdot dx_j}{\sqrt{dx_m \cdot dx_m}} - \frac{\frac{\partial \dot{x}_j}{\partial x_l} \cdot dx_i \cdot dx_j \cdot dx_l}{\left(dx_m \cdot dx_m\right)^{\frac{3}{2}}} \right)$$

$$= \vec{v}_i \cdot \left(\frac{\left(d_{ij} - \omega_{ij}\right) \cdot dx_j}{\sqrt{dx_m \cdot dx_m}} - \frac{\left(d_{jl} - \omega_{jl}\right) \cdot dx_i \cdot dx_j \cdot dx_l}{\left(dx_m \cdot dx_m\right)^{\frac{3}{2}}} \right)$$

Let the x_i directions be principal directions for [d] at point P so that,

$$\left[d\right] = \begin{bmatrix} d_I & 0 & 0 \\ 0 & d_{II} & 0 \\ 0 & 0 & d_{III} \end{bmatrix}$$

and take $dx_1 = 1$, $dx_2 = dx_3 = 0$ so that,

$$\left.\frac{D\vec{\tilde{A}}}{Dt}\right|_{dx_2 = dx_3 = 0} = -\vec{v}_j \cdot \omega_{ji}$$

The same scheme can be followed for the other principal directions so,

$$\frac{D\vec{\tilde{A}}}{Dt} = -\vec{v}_j \cdot \omega_{ji} \cdot dx_i$$

Recall that $\vec{\tilde{A}}$ is a unit length vector at point P and that the last equation was derived for the special case that the x_i directions coincide with the principal directions of [d] at point P.

The remainder of Section V reviews four proposed definitions for stress rate that appear in the Continuum Mechanics literature. They were proposed by,

1. G. Jaumann, 1911
2. C. Truesdell, 1953
3. B. A. Cotter and R. S. Rivlin, 1955
4. J. G. Oldroyd, 1956

The derivations for the definitions are given first. Following this a few examples are presented to give a physical picture of the differences between the definitions.

V.2 JAUMANN STRESS RATE

The derivation for this stress rate definition is based on taking the substantial time derivative for a differential size parallelepiped fixed in the material and with edges aligned with the principal directions of [d]. This derivative is then transformed to spatial coordinates in the current configuration.

As usual, the base vectors in the spatial coordinates are $\vec{v}_i$. Now, at a generic point, determine the unit length base vectors, $\bar{\vec{v}}_i$, that are parallel to the principal directions for the deformation rate, [d], at that point. The stress state s_{ij} in the spatial coordinates is transformed into the orientation of the principal directions of [d] using,

$$\bar{s}_{ij} = s_{kl} \cdot c_{ik} \cdot c_{jl}$$

where, from earlier results,

$$c_{ik} = \bar{\vec{v}} \bullet \vec{v}_k$$

and $\bar{s}_{ij}$ is the stress state in the new orientation. Recalling that $\bar{\vec{v}}_i$ is a set of unit length vectors and the derivation presented in the preceding section for unit length vectors fixed in the material leads to,

$$\frac{D}{Dt}(c_{ik}) = \frac{D\bar{\vec{v}}_i}{Dt} \bullet \vec{v}_k = -\bar{\omega}_{ji} \cdot \bar{\vec{v}}_j \bullet \vec{v}_k = -\bar{\omega}_{ji} \cdot c_{jk}$$

Since ω_{ij} is a second rank tensor the transformation from $[\omega]$ to $[\bar{\omega}]$ is given by,

$$\bar{\omega}_{ji} = \omega_{lm} \cdot c_{jl} \cdot c_{im}$$

Combining the last two equations and recalling that [c] is orthogonal ($[c]^{-1} = [c]^T$) yields,

$$\frac{D}{Dt}(c_{ik}) = -\omega_{lm} \cdot c_{jl} \cdot c_{im} \cdot c_{jk} = -\omega_{lm} \cdot c_{im} \cdot \delta_{lk} = -\omega_{km} \cdot c_{im}$$

Now take the substantial time derivative of $\bar{s}_{ij}$ to obtain,

$$\begin{aligned} \frac{D}{Dt}(\bar{s}_{ij}) &= \frac{D}{Dt}(s_{kl}) \cdot c_{ik} \cdot c_{jl} + s_{kl} \cdot \frac{D}{Dt}(c_{ik}) \cdot c_{jl} + s_{kl} \cdot c_{ik} \cdot \frac{D}{Dt}(c_{jl}) \\ &= \frac{D}{Dt}(s_{kl}) \cdot c_{ik} \cdot c_{jl} - s_{kl} \cdot \omega_{km} \cdot c_{im} \cdot c_{jl} - s_{kl} \cdot c_{ik} \cdot \omega_{lm} \cdot c_{jm} \\ &= c_{ik} \cdot c_{jl} \cdot \left(\frac{D}{Dt}(s_{kl}) - s_{ml} \cdot \omega_{mk} - s_{km} \cdot \omega_{ml} \right) \end{aligned}$$

In order for all the components of this derivative to vanish, the quantities multiplying the coefficient $c_{ik} \cdot c_{jl}$, must vanish. These quantities are defined in the current configuration and they are the basis of Jaumann's stress rate definition, $\dot{s}_{ij}\big|_{JAUMANN}$, which is,

$$\dot{s}_{ij}\big|_{JAUMANN} = \frac{D}{Dt}\left(s_{ij}\right) - s_{mj} \cdot \omega_{mi} - s_{im} \cdot \omega_{mj}$$

or,

$$\left[\dot{s}\right]_{JAUMANN} = \left[\frac{D}{Dt}(s)\right] + [\omega] \cdot [s] - [s] \cdot [\omega]$$

V.3 TRUESDELL STRESS RATE

This definition of stress rate is derived from the Kirchhoff stress tensor. It is noted that the Kirchhoff stress does not have a straightforward physical interpretation so this definition is more a mathematical one than a physical one. The fact that mathematical analyses of elasticity problems related to the reference configuration often use the Kirchhoff stress has probably prompted this definition.

Recall the definition of the Kirchhoff stress is,

$$\mathbf{S}_{\alpha\beta} = s_{ij} \cdot \left\| \left[\frac{\partial x}{\partial X} \right] \right\| \cdot \frac{\partial X_\alpha}{\partial x_i} \cdot \frac{\partial X_\beta}{\partial x_j}$$

or,

$$s_{ik} = \left\| \left[\frac{\partial X}{\partial x} \right] \right\| \cdot \mathbf{S}_{\alpha\beta} \cdot \frac{\partial x_i}{\partial X_\alpha} \cdot \frac{\partial x_k}{\partial X_\beta}$$

This stress rate will be based on $\frac{D}{Dt}\left(\mathbf{S}_{\alpha\beta}\right)$ based on the above equation as follows,

$$\begin{aligned}
\frac{D}{Dt}\left(s_{ij}\right) = {} & \frac{D}{Dt}\left(\left\| \left[\frac{\partial X}{\partial x} \right] \right\| \right) \cdot \mathbf{S}_{\alpha\beta} \cdot \frac{\partial x_i}{\partial X_\alpha} \cdot \frac{\partial x_j}{\partial X_\beta} + \left\| \left[\frac{\partial X}{\partial x} \right] \right\| \cdot \frac{D}{Dt}\left(\mathbf{S}_{\alpha\beta}\right) \frac{\partial x_i}{\partial X_\alpha} \cdot \frac{\partial x_j}{\partial X_\beta} \\
& + \left\| \left[\frac{\partial X}{\partial x} \right] \right\| \cdot \mathbf{S}_{\alpha\beta} \cdot \frac{D}{Dt}\left(\frac{\partial x_i}{\partial X_\alpha} \right) \cdot \frac{\partial x_j}{\partial X_\beta} + \left\| \left[\frac{\partial X}{\partial x} \right] \right\| \cdot \mathbf{S}_{\alpha\beta} \cdot \frac{\partial x_i}{\partial X_\alpha} \cdot \frac{D}{Dt}\left(\frac{\partial x_j}{\partial X_\beta} \right)
\end{aligned}$$

$$\begin{aligned}
= {} & -\frac{\partial \dot{x}_k}{\partial x_k} \cdot s_{ij} + \left\| \left[\frac{\partial X}{\partial x} \right] \right\| \cdot \frac{D}{Dt}\left(\mathbf{S}_{\alpha\beta}\right) \frac{\partial x_i}{\partial X_\alpha} \cdot \frac{\partial x_j}{\partial X_\beta} + \left\| \left[\frac{\partial X}{\partial x} \right] \right\| \cdot \mathbf{S}_{\alpha\beta} \cdot \frac{\partial \dot{x}_i}{\partial x_l} \cdot \frac{\partial x_l}{\partial X_\alpha} \cdot \frac{\partial x_j}{\partial X_\beta} \\
& + \left\| \left[\frac{\partial X}{\partial x} \right] \right\| \cdot \mathbf{S}_{\alpha\beta} \cdot \frac{\partial \dot{x}_j}{\partial x_l} \cdot \frac{\partial x_i}{\partial X_\alpha} \cdot \frac{\partial x_l}{\partial X_\beta}
\end{aligned}$$

$$= \left\| \left[\frac{\partial X}{\partial x} \right] \right\| \cdot \frac{D}{Dt}\left(\mathbf{S}_{\alpha\beta}\right) \frac{\partial x_i}{\partial X_\alpha} \cdot \frac{\partial x_j}{\partial X_\beta} - \frac{\partial \dot{x}_k}{\partial x_k} \cdot s_{ij} + s_{lj} \cdot \frac{\partial \dot{x}_i}{\partial x_l} + s_{il} \cdot \frac{\partial \dot{x}_j}{\partial x_l}$$

so that,

$$\frac{D}{Dt}\left(\mathbf{S}_{\alpha\beta}\right) = \left\| \left[\frac{\partial X}{\partial x} \right] \right\| \cdot \frac{\partial X_\alpha}{\partial x_i} \cdot \frac{\partial X_\beta}{\partial x_j} \cdot \left(\frac{D}{Dt}\left(s_{ij}\right) + \frac{\partial \dot{x}_k}{\partial x_k} \cdot s_{ij} - s_{lj} \cdot \frac{\partial \dot{x}_i}{\partial x_l} - s_{il} \cdot \frac{\partial \dot{x}_j}{\partial x_l} \right)$$

The stress rate is defined to be zero when all of the components of $\frac{D}{Dt}\left(\mathbf{s}_{\alpha\beta}\right)$ vanish. This provides an acceptable stress rate, $\dot{s}_{ij}\big|_{TRUESDELL}$, as,

$$\dot{s}_{ij}\Big|_{TRUESDELL} = \frac{D}{Dt}\left(s_{ij}\right) + \frac{\partial \dot{x}_k}{\partial x_k} \cdot s_{ij} - s_{lj} \cdot \frac{\partial \dot{x}_i}{\partial x_l} - s_{ilj} \cdot \frac{\partial \dot{x}_j}{\partial x_l}$$

or,

$$\left[\dot{s}\right]_{TRUESDELL} = \left[\frac{D}{Dt}\left(s\right)\right] + d_{kk} \cdot \left[s\right] - \left[\frac{\partial \dot{x}}{\partial x}\right] \cdot \left[s\right] - \left[s\right] \cdot \left[\frac{\partial \dot{x}}{\partial x}\right]^T$$

V.4 COTTER-RIVLIN STRESS RATE

This definition for stress rate is similar to the Jaumann definition. In this case the new "base vectors", $\tilde{\vec{v}}_i$, are fixed in the material and referred to the current configuration. Initially these vectors coincide with the spatial base vectors so they are initially unit length vectors but their length and orientation change with time. That is, the transformation between this material coordinate system and the current configuration is initially $c_{ij} = \delta_{ij}$ but it is not constant in time. This transformation is,

$$c_{ik} = \tilde{\vec{v}}_i \bullet \vec{v}_k$$

so that,

$$\frac{D}{Dt}(c_{ik}) = \frac{D}{Dt}\left(\tilde{\vec{v}}_i\right) \bullet \vec{v}_k = \frac{\partial \dot{x}_i}{\partial x_j} \cdot \tilde{\vec{v}}_j \bullet \vec{v}_k = \frac{\partial \dot{x}_i}{\partial x_j} \cdot c_{jk}$$

Let $\tilde{s}_{kl}$ be the transformed stress and take its substantial derivative to obtain,

$$\begin{aligned}
\frac{D}{Dt}(\tilde{s}_{kl}) &= \frac{D}{Dt}\left(s_{ij}\right) \cdot c_{ki} \cdot c_{lj} + s_{ij} \cdot \frac{D}{Dt}(c_{ki}) \cdot c_{lj} + s_{ij} \cdot c_{ki} \cdot \frac{D}{Dt}\left(c_{lj}\right) \\
&= \frac{D}{Dt}\left(s_{ij}\right) \cdot c_{ki} \cdot c_{lj} + s_{ij} \cdot \frac{\partial \dot{x}_k}{\partial x_m} \cdot c_{mi} \cdot c_{lj} + s_{ij} \cdot \frac{\partial \dot{x}_l}{\partial x_m} \cdot c_{ki} \cdot c_{mj}
\end{aligned}$$

Since, at the instant of evaluation, $c_{ij} = \delta_{ij}$,

$$\frac{D}{Dt}(\tilde{s}_{kl}) = \frac{D}{Dt}(s_{kl}) + s_{ml} \cdot \frac{\partial \dot{x}_m}{\partial x_k} + s_{km} \cdot \frac{\partial \dot{x}_l}{\partial x_m}$$

This last expression is defined as the Cotter-Rivlin stress rate, $\dot{s}_{ij}\big|_{\text{COTTER-RIVLIN}}$, so,

$$\begin{aligned}
\dot{s}_{ij}\big|_{\text{COTTER-RIVLIN}} &= \frac{D}{Dt}\left(s_{ij}\right) + s_{mj} \cdot \frac{\partial \dot{x}_m}{\partial x_i} + s_{im} \cdot \frac{\partial \dot{x}_j}{\partial x_m} \\
&= \frac{D}{Dt}\left(s_{ij}\right) + s_{mj} \cdot (d_{mi} - \omega_{mi}) + s_{im} \cdot \left(d_{jm} - \omega_{jm}\right)
\end{aligned}$$

or,

$$\begin{aligned}
\left[\dot{s}\right]_{\text{COTTER-RIVLIN}} &= \left[\frac{D}{Dt}(s)\right] + \left[\frac{\partial \dot{x}}{\partial x}\right]^T \cdot [s] + [s] \cdot \left[\frac{\partial \dot{x}}{\partial x}\right]^T \\
&= \left[\frac{D}{Dt}(s)\right] + 2 \cdot [d] \cdot [s] + [\omega] \cdot [s] + [s] \cdot [\omega]
\end{aligned}$$

V.5 OLDROYD STRESS RATE

This definition is similar to the Truesdell definition as it starts with a stress, $\mathbf{S}_{\alpha\beta}$, referred to the reference configuration. The physical basis for the stress is not obvious from its definition which is,

$$\mathbf{S}_{\alpha\beta} = s_{ij} \cdot \frac{\partial X_\alpha}{\partial x_i} \cdot \frac{\partial X_\beta}{\partial x_j}$$

so that,

$$s_{ij} = \mathbf{S}_{\alpha\beta} \cdot \frac{\partial x_i}{\partial X_\alpha} \cdot \frac{\partial x_j}{\partial X_\beta}$$

Take the substantial time derivative of s_{ij} to find,

$$\begin{aligned}
\frac{D}{Dt}(s_{ij}) &= \frac{D}{Dt}(\mathbf{S}_{\alpha\beta}) \cdot \frac{\partial x_i}{\partial X_\alpha} \cdot \frac{\partial x_j}{\partial X_\beta} + \mathbf{S}_{\alpha\beta} \cdot \frac{D}{Dt}\left(\frac{\partial x_i}{\partial X_\alpha}\right) \cdot \frac{\partial x_j}{\partial X_\beta} + \mathbf{S}_{\alpha\beta} \cdot \frac{\partial x_i}{\partial X_\alpha} \cdot \frac{D}{Dt}\left(\frac{\partial x_j}{\partial X_\beta}\right) \\
&= \frac{D}{Dt}(\mathbf{S}_{\alpha\beta}) \cdot \frac{\partial x_i}{\partial X_\alpha} \cdot \frac{\partial x_j}{\partial X_\beta} + \mathbf{S}_{\alpha\beta} \cdot \frac{\partial \dot{x}_i}{\partial x_l} \cdot \frac{\partial x_l}{\partial X_\alpha} \cdot \frac{\partial x_j}{\partial X_\beta} + \mathbf{S}_{\alpha\beta} \cdot \frac{\partial \dot{x}_j}{\partial x_l} \cdot \frac{\partial x_i}{\partial X_\alpha} \cdot \frac{\partial x_l}{\partial X_\beta} \\
&= \frac{D}{Dt}(\mathbf{S}_{\alpha\beta}) \cdot \frac{\partial x_i}{\partial X_\alpha} \cdot \frac{\partial x_j}{\partial X_\beta} + s_{lj} \cdot \frac{\partial \dot{x}_i}{\partial x_l} + s_{il} \cdot \frac{\partial \dot{x}_j}{\partial x_l}
\end{aligned}$$

so that,

$$\frac{D}{Dt}(\mathbf{S}_{\alpha\beta}) = \frac{\partial X_\alpha}{\partial x_k} \cdot \frac{\partial X_\beta}{\partial x_m} \cdot \left(\frac{D}{Dt}(s_{km}) - s_{lm} \cdot \frac{\partial \dot{x}_k}{\partial x_l} - s_{kl} \cdot \frac{\partial \dot{x}_m}{\partial x_l}\right)$$

The Oldroyd stress rate, $\dot{s}_{ij}\big|_{OLDROYD}$, vanishes when all the components of $\frac{D}{Dt}(\mathbf{S}_{\alpha\beta})$ vanish and it is,

$$\dot{s}_{km}\big|_{OLDROYD} = \frac{D}{Dt}(s_{km}) - s_{lm} \cdot \frac{\partial \dot{x}_k}{\partial x_l} - s_{kl} \cdot \frac{\partial \dot{x}_m}{\partial x_l}$$

or,

$$[\dot{s}]_{OLDROYD} = \left[\frac{D}{Dt}(s)\right] - \left[\frac{\partial \dot{x}}{\partial x}\right] \cdot [s] - [s] \cdot \left[\frac{\partial \dot{x}}{\partial x}\right]^T$$

Note that,

$$\dot{s}_{ij}\Big|_{\text{TRUESDELL}} = \dot{s}_{ij}\Big|_{\text{OLDROYD}} + s_{ij} \cdot \frac{\partial \dot{x}_k}{\partial x_k}$$

V.6 ILLUSTRATIVE EXAMPLES

EXAMPLE 1 - ROTATION UNDER TENSION (revisiting earlier problem)

In Section V.1 the case of a rod under tension, σ, in the x_1-x_2 plane rotating at an angular rate of Ω about the x_3 axis was studied to show that $\frac{D}{Dt}(s_{ij})$ is not a valid stress rate for use in a constitutive equation. This illustrative example gives the results of finding the stress rates according to the four definitions reviewed above. The motion for this case is,

$$\dot{x}_1 = -\Omega \cdot x_2$$

$$\dot{x}_2 = \Omega \cdot x_1$$

so that,

$$[d] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$[\omega] = \begin{bmatrix} 0 & \Omega & 0 \\ -\Omega & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and from the earlier consideration,

$$[s] = \begin{bmatrix} \frac{1}{2} \cdot \sigma \cdot (1 + \cos(2 \cdot \Omega \cdot t)) & \frac{1}{2} \cdot \sigma \cdot \sin(2 \cdot \Omega \cdot t) & 0 \\ \frac{1}{2} \cdot \sigma \cdot \sin(2 \cdot \Omega \cdot t) & \frac{1}{2} \cdot \sigma \cdot (1 + \cos(2 \cdot \Omega \cdot t)) & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\frac{D s_{ij}}{Dt} = \begin{bmatrix} -\sigma \cdot \Omega \cdot \sin(\Omega \cdot t) & \sigma \cdot \Omega \cdot \cos(\Omega \cdot t) & 0 \\ \sigma \cdot \Omega \cdot \cos(\Omega \cdot t) & \sigma \cdot \Omega \cdot \sin(\Omega \cdot t) & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The results for this illustrative example are given below.

$$\dot{s}_{ij}\Big|_{\text{JAUMANN}} = \begin{bmatrix} -\sigma\cdot\Omega\cdot\sin(\Omega\cdot t) & \sigma\cdot\Omega\cdot\cos(\Omega\cdot t) & 0 \\ \sigma\cdot\Omega\cdot\cos(\Omega\cdot t) & \sigma\cdot\Omega\cdot\sin(\Omega\cdot t) & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$-\begin{bmatrix} 0 & -\Omega & 0 \\ \Omega & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\cdot\begin{bmatrix} \frac{1}{2}\cdot\sigma\cdot(1+\cos(2\cdot\Omega\cdot t)) & \frac{1}{2}\cdot\sigma\cdot\sin(2\cdot\Omega\cdot t) & 0 \\ \frac{1}{2}\cdot\sigma\cdot\sin(2\cdot\Omega\cdot t) & \frac{1}{2}\cdot\sigma\cdot(1-\cos(2\cdot\Omega\cdot t)) & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$-\begin{bmatrix} \frac{1}{2}\cdot\sigma\cdot(1+\cos(2\cdot\Omega\cdot t)) & \frac{1}{2}\cdot\sigma\cdot\sin(2\cdot\Omega\cdot t) & 0 \\ \frac{1}{2}\cdot\sigma\cdot\sin(2\cdot\Omega\cdot t) & \frac{1}{2}\cdot\sigma\cdot(1-\cos(2\cdot\Omega\cdot t)) & 0 \\ 0 & 0 & 0 \end{bmatrix}\cdot\begin{bmatrix} 0 & \Omega & 0 \\ -\Omega & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\dot{s}_{ij}\Big|_{\text{OLDROYD}} = \begin{bmatrix} -\sigma\cdot\Omega\cdot\sin(\Omega\cdot t) & \sigma\cdot\Omega\cdot\cos(\Omega\cdot t) & 0 \\ \sigma\cdot\Omega\cdot\cos(\Omega\cdot t) & \sigma\cdot\Omega\cdot\sin(\Omega\cdot t) & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$-\begin{bmatrix} 0 & -\Omega & 0 \\ \Omega & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\cdot\begin{bmatrix} \frac{1}{2}\cdot\sigma\cdot(1+\cos(2\cdot\Omega\cdot t)) & \frac{1}{2}\cdot\sigma\cdot\sin(2\cdot\Omega\cdot t) & 0 \\ \frac{1}{2}\cdot\sigma\cdot\sin(2\cdot\Omega\cdot t) & \frac{1}{2}\cdot\sigma\cdot(1-\cos(2\cdot\Omega\cdot t)) & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$-\begin{bmatrix} \frac{1}{2}\cdot\sigma\cdot(1+\cos(2\cdot\Omega\cdot t)) & \frac{1}{2}\cdot\sigma\cdot\sin(2\cdot\Omega\cdot t) & 0 \\ \frac{1}{2}\cdot\sigma\cdot\sin(2\cdot\Omega\cdot t) & \frac{1}{2}\cdot\sigma\cdot(1-\cos(2\cdot\Omega\cdot t)) & 0 \\ 0 & 0 & 0 \end{bmatrix}\cdot\begin{bmatrix} 0 & \Omega & 0 \\ -\Omega & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Since all d_{ij} vanish,

$$\dot{s}_{ij}\Big|_{\text{COTTER-RIVLIN}} = \dot{s}_{ij}\Big|_{\text{JAUMANN}} \quad \text{and} \quad \dot{s}_{ij}\Big|_{\text{TRUESDELL}} = \dot{s}_{ij}\Big|_{\text{OLDROYD}}$$

As expected, all four stress rates vanish for this case.

EXAMPLE 2 - RECTANGULAR MOTION

The equations governing this motion are ($A \cdot t > -1$, $B \cdot t > 1$, $C \cdot t > 1$),

$$x_1 = X_1 \cdot (1 + A \cdot t) \qquad X_1 = \frac{x_1}{1 + A \cdot t} \qquad \dot{x}_1 = A \cdot X_1 = \frac{A \cdot x_1}{1 + A \cdot t}$$

$$x_2 = X_2 \cdot (1 - B \cdot t) \qquad X_2 = \frac{x_2}{1 - B \cdot t} \qquad \dot{x}_2 = -B \cdot X_2 = \frac{B \cdot x_2}{1 - B \cdot t}$$

$$x_3 = X_3 \cdot (1 - C \cdot t) \qquad X_3 = \frac{x_3}{1 - C \cdot t} \qquad \dot{x}_3 = -C \cdot X_3 = \frac{C \cdot x_3}{1 - C \cdot t}$$

The stress state has only normal components, is constant in time and given by,

$$[s] = \begin{bmatrix} s_1 & 0 & 0 \\ 0 & s_2 & 0 \\ 0 & 0 & s_3 \end{bmatrix}$$

For the motion,

$$[d] = \begin{bmatrix} \frac{A}{1 + A \cdot t} & 0 & 0 \\ 0 & \frac{-B}{1 - B \cdot t} & 0 \\ 0 & 0 & \frac{-C}{1 - C \cdot t} \end{bmatrix}$$

$$\frac{\partial \dot{x}_k}{\partial x_k} = d_{kk} = \frac{A}{1 + A \cdot t} - \frac{B}{1 - B \cdot t} - \frac{C}{1 - C \cdot t}$$

$$\left[\frac{\partial \dot{x}_i}{\partial x_j}\right] = \begin{bmatrix} \frac{A}{1 + A \cdot t} & 0 & 0 \\ 0 & \frac{-B}{1 - B \cdot t} & 0 \\ 0 & 0 & \frac{-C}{1 - C \cdot t} \end{bmatrix}$$

$$[\omega] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\left[\frac{D}{Dt}\left(s_{ij}\right)\right] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The stress rates are,

$$\dot{s}_{ij}\Big|_{JAUMANN} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\dot{s}_{ij}\Big|_{\mathrm{TRUESDELL}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \left(\frac{A}{1+A\cdot t} - \frac{B}{1-B\cdot t} - \frac{C}{1-C\cdot t}\right)\cdot\begin{bmatrix} s_1 & 0 & 0 \\ 0 & s_2 & 0 \\ 0 & 0 & s_3 \end{bmatrix}$$

$$-\begin{bmatrix} \frac{A}{1+A\cdot t} & 0 & 0 \\ 0 & \frac{-B}{1-B\cdot t} & 0 \\ 0 & 0 & \frac{-C}{1-C\cdot t} \end{bmatrix}\cdot\begin{bmatrix} s_1 & 0 & 0 \\ 0 & s_2 & 0 \\ 0 & 0 & s_3 \end{bmatrix}$$

$$-\begin{bmatrix} s_1 & 0 & 0 \\ 0 & s_2 & 0 \\ 0 & 0 & s_3 \end{bmatrix}\cdot\begin{bmatrix} \frac{A}{1+A\cdot t} & 0 & 0 \\ 0 & \frac{-B}{1-B\cdot t} & 0 \\ 0 & 0 & \frac{-C}{1-C\cdot t} \end{bmatrix}$$

$$= \begin{bmatrix} \left(-\frac{A}{1+A\cdot t} - \frac{B}{1-B\cdot t} - \frac{C}{1-C\cdot t}\right)\cdot s_1 & 0 & 0 \\ 0 & \left(\frac{A}{1+A\cdot t} + \frac{B}{1-B\cdot t} - \frac{C}{1-C\cdot t}\right)\cdot s_2 & 0 \\ 0 & 0 & \left(\frac{A}{1+A\cdot t} - \frac{B}{1-B\cdot t} + \frac{C}{1-C\cdot t}\right)\cdot s_3 \end{bmatrix}$$

$$\dot{s}_{ij}\Big|_{\text{COTTER-RIVLIN}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} \dfrac{A}{1+A\cdot t} & 0 & 0 \\ 0 & \dfrac{-B}{1-B\cdot t} & 0 \\ 0 & 0 & \dfrac{-C}{1-C\cdot t} \end{bmatrix} \cdot \begin{bmatrix} s_1 & 0 & 0 \\ 0 & s_2 & 0 \\ 0 & 0 & s_3 \end{bmatrix}$$

$$+ \begin{bmatrix} s_1 & 0 & 0 \\ 0 & s_2 & 0 \\ 0 & 0 & s_3 \end{bmatrix} \cdot \begin{bmatrix} \dfrac{A}{1+A\cdot t} & 0 & 0 \\ 0 & \dfrac{-B}{1-B\cdot t} & 0 \\ 0 & 0 & \dfrac{-C}{1-C\cdot t} \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{2\cdot A\cdot s_1}{1+A\cdot t} & 0 & 0 \\ 0 & \dfrac{-2\cdot B\cdot s_2}{1-B\cdot t} & 0 \\ 0 & 0 & \dfrac{-2\cdot C\cdot s_3}{1-C\cdot t} \end{bmatrix}$$

$$\dot{s}_{ij}\Big|_{\text{OLDROYD}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} \frac{A}{1+A\cdot t} & 0 & 0 \\ 0 & \frac{-B}{1-B\cdot t} & 0 \\ 0 & 0 & \frac{-C}{1-C\cdot t} \end{bmatrix} \cdot \begin{bmatrix} s_1 & 0 & 0 \\ 0 & s_2 & 0 \\ 0 & 0 & s_3 \end{bmatrix}$$

$$- \begin{bmatrix} s_1 & 0 & 0 \\ 0 & s_2 & 0 \\ 0 & 0 & s_3 \end{bmatrix} \cdot \begin{bmatrix} \frac{A}{1+A\cdot t} & 0 & 0 \\ 0 & \frac{-B}{1-B\cdot t} & 0 \\ 0 & 0 & \frac{-C}{1-C\cdot t} \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{2\cdot A\cdot s_1}{1+A\cdot t} & 0 & 0 \\ 0 & \frac{2\cdot B\cdot s_2}{1-B\cdot t} & 0 \\ 0 & 0 & \frac{2\cdot C\cdot s_3}{1-C\cdot t} \end{bmatrix}$$

It is instructive to consider several special cases of the solution given above for the four stress rates.

If $A > 0$, $B = C = 0$ and $s_1 > 0$, $s_2 = s_3 = 0$ then there is uniaxial straining with a constant axial stress and the only nontrivial values of the stress rates are,

$$\dot{s}_{11}\Big|_{\text{JAUMANN}} = 0$$

$$\dot{s}_{11}\Big|_{\text{TRUESDELL}} = -\frac{A\cdot s_1}{1+A\cdot t}$$

$$\dot{s}_{11}\Big|_{\text{COTTER-RIVLIN}} = \frac{2\cdot A\cdot s_1}{1+A\cdot t}$$

$$\dot{s}_{11}\Big|_{\text{OLDROYD}} = -\frac{2\cdot A\cdot s_1}{1+A\cdot t}$$

If $A > 0$, $B = C = \frac{1}{2}\cdot A$ and $s_1 > 0$, $s_2 = s_3 = 0$ then at $t = 0$ this is extension under uniaxial tension with initially constant volume motion and the only nontrivial values of the stress rates are,

$$\dot{s}_{11}\Big|_{JAUMANN} = 0$$

$$\dot{s}_{11}\Big|_{TRUESDELL} = -\frac{2 \cdot A \cdot s_1}{1 + A \cdot t}$$

$$\dot{s}_{11}\Big|_{COTTER-RIVLIN} = \frac{2 \cdot A \cdot s_1}{1 + A \cdot t}$$

$$\dot{s}_{11}\Big|_{OLDROYD} = -\frac{2 \cdot A \cdot s_1}{1 + A \cdot t}$$

If $A > 0$, $B = C = -A$ and $s_1 = s_2 = s_3 = \sigma$ then there is uniform expansion with a constant hydrostatic stress and the only nontrivial values of the stress rates are,

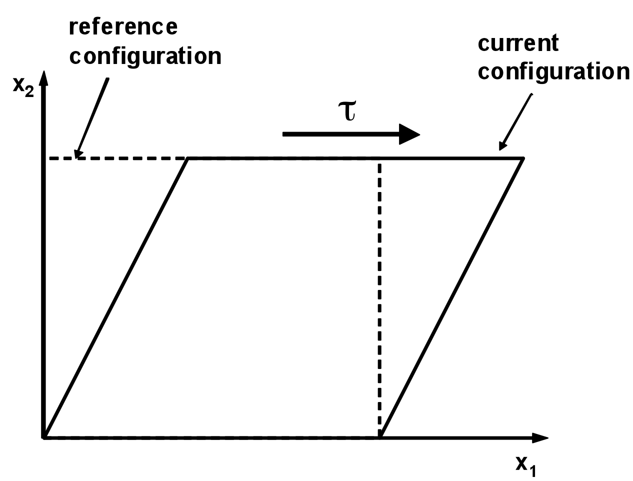
$$\dot{s}_{11}\Big|_{JAUMANN} = \dot{s}_{22}\Big|_{JAUMANN} = \dot{s}_{33}\Big|_{JAUMANN} = 0$$

$$\dot{s}_{11}\Big|_{TRUESDELL} = \dot{s}_{22}\Big|_{TRUESDELL} = \dot{s}_{33}\Big|_{TRUESDELL} = \frac{A \cdot \sigma}{1 + A \cdot t}$$

$$\dot{s}_{11}\Big|_{COTTER-RIVLIN} = \dot{s}_{22}\Big|_{COTTER-RIVLIN} = \dot{s}_{33}\Big|_{COTTER-RIVLIN} = \frac{2 \cdot A \cdot \sigma}{1 + A \cdot t}$$

$$\dot{s}_{11}\Big|_{OLDROYD} = \dot{s}_{22}\Big|_{OLDROYD} = \dot{s}_{33}\Big|_{OLDROYD} = -\frac{2 \cdot A \cdot \sigma}{1 + A \cdot t}$$

EXAMPLE 3 - SHEAR MOTION



The above sketch shows the shearing motion considered for this example. The equations describing this motion are,

$$x_1 = X_1 + M \cdot t \cdot X_2 \qquad X_1 = x_1 - M \cdot t \cdot x_2 \qquad \dot{x}_1 = M \cdot X_2 = M \cdot x_2$$
$$x_2 = X_2 \qquad X_2 = x_2 \qquad \dot{x}_2 = 0$$
$$x_3 = X_3 \qquad X_3 = x_3 \qquad \dot{x}_3 = 0$$

and the stress state is constant in time and given by,

$$[s] = \begin{bmatrix} 0 & \tau & 0 \\ \tau & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and,

$$\left[\frac{D}{Dt}(s_{ij})\right] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The motion yields the following results,

$$\left[\frac{\partial \dot{x}_i}{\partial x_j}\right] = \begin{bmatrix} 0 & M & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$[d] = \begin{bmatrix} 0 & \frac{1}{2} \cdot M & 0 \\ \frac{1}{2} \cdot M & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$[\omega] = \begin{bmatrix} 0 & -\frac{1}{2} \cdot M & 0 \\ \frac{1}{2} \cdot M & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The four stress rates are now calculated.

$$\dot{s}_{ij}\Big|_{JAUMANN} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & \frac{1}{2}\cdot M & 0 \\ -\frac{1}{2}\cdot M & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & \tau & 0 \\ \tau & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & \tau & 0 \\ \tau & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & -\frac{1}{2}\cdot M & 0 \\ \frac{1}{2}\cdot M & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -\tau\cdot M & 0 & 0 \\ 0 & \tau\cdot M & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\dot{s}_{ij}\Big|_{TRUESDELL} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & M & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & \tau & 0 \\ \tau & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & \tau & 0 \\ \tau & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 & 0 \\ M & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} -2\cdot\tau\cdot M & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\dot{s}_{ij}\Big|_{COTTER-RIVLIN} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ M & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & \tau & 0 \\ \tau & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & \tau & 0 \\ \tau & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & M & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2\cdot\tau\cdot M & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\dot{s}_{ij}\Big|_{OLDROYD} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & M & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & \tau & 0 \\ \tau & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & \tau & 0 \\ \tau & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 & 0 \\ M & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -2\cdot\tau\cdot M & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

VI THERMODYNAMIC CONSIDERATIONS

VI.1 INTRODUCTION

A requirement from physical science is that any theory must conform to the general results from Thermodynamics. The purpose of Section VI is to tie the results given above to Classical Thermodynamics and to determine any constraints that must be imposed on constitutive equations in order for them to conform to Classical Thermodynamics. A brief review of Classical Thermodynamics is in an appendix included at the end of this work.

In the case of a Continuum Mechanics formulation the system is a fixed mass particle so that the stresses and strains may be considered uniform. The first essential part of the Appendix concerns the First Law of Thermodynamics for fixed mass systems which is expressed as,

$$\dot{q} = \dot{E} + \dot{W}$$

where,

$\dot{q}$ = time rate of heat flow into the system from its exterior
= $T \cdot \dot{s}$ for a reversible process
$\dot{E}$ = time rate of change of internal energy
$\dot{W}$ = time rate of work being done by the system on its exterior

In addition, define,

T = absolute temperature
$\dot{s}$ = time rate of change of entropy
ρ = mass density

When a specific constitutive equation is considered, it is often possible to determine $\dot{W}$ explicitly. Substituting $\dot{W}$ into the first law and solving for $\dot{s}$ yields an equation whose validity must be determined. The condition that s be a perfect differential (i.e. s is a property dependent only on the state of the material) leads to a condition that must be satisfied by E. When this condition is satisfied then the Inequality of Clausis is valid and it becomes a way of expressing the Second Law of Thermodynamics as,

$$\oint \frac{dq}{T} \equiv \oint ds \geq 0$$

where the inequality becomes the equality only for a reversible cycle.

A special, important result from Thermodynamics is that an equilibrium condition is defined. This is accomplished by considering an isolated system ($\dot{q}$ and $\dot{W}$ are both zero) and showing, based on the Inequality of Clausis, that any spontaneous change of the thermodynamic properties will result in an increase in entropy of the system. Equilibrium is defined as a stable state where no spontaneous change occurs in an isolated system and it implies that E will be a minimum in this state. For small deviations in the thermodynamic properties from the state being considered the conditions,

$$\delta E = 0 \quad \text{and} \quad \delta^2 E > 0$$

must be satisfied where δE and $\delta^2 E$ are the first and second variations of E. It is noted that the usual stability calculations concerning buckling of beams and other structures is not covered by these considerations although the thermodynamic results can be extended to cover structural stability.

Several common, elementary, constitutive equations are reviewed in the section below. In each case, the expressions for E and $\dot{s}$ are determined as well as investigating the conditions for an equilibrium state. In each case, the derivation is given in spatial coordinates. These reviews show the kind of restrictions thermodynamics imposes on constitutive equations.

VI.2 SELECTED, ILLUSTRATIVE, CONSTITUTIVE EQUATIONS

LINEAR THERMOELASTICITY

The constitutive equation relates the strains, e_{ij}, stresses, s_{ij}, and absolute temperature, T, as,

$$\sigma_{ij} = \rho \cdot \hat{\lambda} \cdot \left(e_{kk} - 3 \cdot \alpha \cdot \left(T - T_O\right)\right) \cdot \delta_{ij} + 2 \cdot \rho \cdot \hat{G} \cdot \left(e_{ij} - \alpha \cdot \left(T - T_O\right) \cdot \delta_{ij}\right)$$

where in terms of Young's modulus, $\hat{E}$, and Poisson's ratio, ν,

$$\rho \cdot \hat{\lambda} = \frac{\nu \cdot \hat{E}}{(1 + \nu) \cdot (1 - 2 \cdot \nu)}$$

$$\rho \cdot \hat{G} = \frac{\hat{E}}{2 \cdot (1 + \nu)}$$

and

α = thermal coefficient of linear expansion
ρ = mass density, a function of e_{ij} and T
T_O = a constant reference temperature

The parameters $\hat{\lambda}$, $\hat{G}$ and α are constants so that, up to this point, E and ν are functions of e_{ij} and T.

The thermodynamic system under consideration is a particle whose mass is constant. The rate of work done by this particle is,

$$\dot{W} = -\frac{\sigma_{ij} \cdot \dot{e}_{ij}}{\rho}$$

Now assume the internal energy, E, is a function of e_{ij} and T so that,

$$\dot{E} = \frac{\partial E}{\partial e_{ij}} \cdot \dot{e}_{ij} + \frac{\partial E}{\partial T} \cdot \dot{T}$$

and the Thermodynamic First Law gives,

$$\dot{q} = \dot{E} + \dot{W} = \left(\frac{\partial E}{\partial e_{ij}} - \frac{\sigma_{ij}}{\rho} \right) \cdot \dot{e}_{ij} + \frac{\partial E}{\partial T} \cdot \dot{T}$$

The entropy production rate, $\dot{s}$, is,

$$\dot{s} = \frac{\dot{q}}{T} = \frac{1}{T} \cdot \left(\frac{\partial E}{\partial e_{ij}} - \frac{\sigma_{ij}}{\rho} \right) \cdot \dot{e}_{ij} + \frac{1}{T} \cdot \frac{\partial E}{\partial T} \cdot \dot{T}$$

In order that s be a state function, the following condition must be satisfied,

$$\frac{\partial}{\partial T}\left(\frac{1}{T} \cdot \left(\frac{\partial E}{\partial e_{ij}} - \frac{\sigma_{ij}}{\rho} \right) \right) = \frac{\partial}{\partial e_{ij}}\left(\frac{1}{T} \cdot \frac{\partial E}{\partial T} \right)$$

which gives,

$$\frac{\partial E}{\partial e_{ij}} = \frac{\sigma_{ij}}{\rho} - T \cdot \frac{\partial}{\partial T}\left(\frac{\sigma_{ij}}{\rho} \right)$$

When the thermoelasticity constitutive equation given above is substituted into this condition, the result is,

$$\frac{\partial E}{\partial e_{ij}} = \hat{\lambda} \cdot e_{kk} \cdot \delta_{ij} + 2 \cdot \hat{G} \cdot e_{ij} + \left(3 \cdot \hat{\lambda} + 2 \cdot \hat{G}\right) \alpha \cdot T_O \cdot \delta_{ij}$$

Integration of $\frac{\partial E}{\partial e_{ij}}$ gives,

$$E = \tfrac{1}{2} \cdot \hat{\lambda} \cdot (e_{kk})^2 + \hat{G} \cdot e_{ij} \cdot e_{ij} + (3 \cdot \hat{\lambda} + 2 \cdot \hat{G})\alpha \cdot T_O \cdot e_{kk} + H(T)$$

then,

$$\dot{q} = (3 \cdot \hat{\lambda} + 2 \cdot \hat{G})\alpha \cdot T \cdot \dot{e}_{kk} + \frac{\partial H(T)}{\partial T} \cdot \dot{T}$$

$$\dot{s} = (3 \cdot \hat{\lambda} + 2 \cdot \hat{G})\alpha \cdot \dot{e}_{kk} + \frac{\partial H(T)}{\partial T} \cdot \frac{\dot{T}}{T}$$

In order to relate H(T) to a physically familiar quantity, note that T is an independent thermodynamic property and when the strain rates are zero the value of $\dot{q}$ is,

$$\dot{q}\Big|_{\dot{e}_{ij}=0} = \frac{\partial H(T)}{\partial T} \cdot \dot{T} = c_V \cdot \dot{T}$$

where c_V is the specific heat at constant volume and assumed to be constant. Then. H(T) may be written as,

$$H(T) = c_V \cdot (T - T_1)$$

where T_1 is a constant of integration. Consequently, E may be written as,

$$E = \tfrac{1}{2} \cdot \hat{\lambda} \cdot (e_{kk})^2 + \hat{G} \cdot e_{ij} \cdot e_{ij} + c_V \cdot (T - T_1)$$

and,

$$\dot{q} = (3 \cdot \hat{\lambda} + 2 \cdot \hat{G})\alpha \cdot T \cdot \dot{e}_{kk} + c_V \cdot \dot{T}$$

$$\dot{s} = (3 \cdot \hat{\lambda} + 2 \cdot \hat{G})\alpha \cdot \dot{e}_{kk} + c_V \cdot \frac{\dot{T}}{T}$$

In the case of linear thermoelasticity the constitutive equation is linearized with respect to the strains and the temperature. This process causes the value of ρ to be a constant in the constitutive equation and then $\hat{E}$ and ν are also constants in accordance with the usual thermoelastic theory.

When this constant mass thermoelastic system is isolated, the first variation of E, δE, vanishes since $\dot{E}$ vanishes and the second variation, $\delta^2 E$, is positive definite as the quadratic quantity $\frac{1}{2} \cdot \lambda \cdot (e_{kk})^2 + G \cdot e_{ij} \cdot e_{ij}$ is positive definite in e_{ij}. Consequently, the system is stable.

The temperature, T, is the absolute temperature in the above derivation. When the constitutive equation is used in problem solving, it is common to replace $(T-T_O)$ with a temperature that is not a true thermodynamic temperature (e.g. degrees Celsius).

LINEAR, VISCOUS, COMPRESSIBLE NEWTONIAN FLUID

The constitutive equation for this fluid may be written in terms of the stress, σ_{ij}, the strain, e_{ij}, the strain rate, $\dot{e}_{ij}$, and absolute temperature, T, in the form,

$$\sigma_{ij} = \rho \cdot \widetilde{\lambda} \cdot \dot{e}_{kk} \cdot \delta_{ij} + 2 \cdot \rho \cdot \overleftrightarrow{\mu} \cdot \dot{e}_{ij} + \rho \cdot \overleftrightarrow{C} \cdot \left(e_{kk} - 3 \cdot \alpha \cdot \left(T - T_O\right)\right) \cdot \delta_{ij}$$

where,

$\overleftrightarrow{\lambda}$ and $\overleftrightarrow{\mu}$	= volumetric and shear viscosities, constant material properties
$\overleftrightarrow{C}$	= elastic compressibility, constant material property
α	= thermal coefficient of linear expansion, a constant
ρ	= mass density, a function of e_{kk} and T
T_O	= a constant reference temperature

This case of a fluid introduces new considerations to the determination of the internal energy and the entropy functions. The presence of a viscosity implies that there is a dissipation of energy *within* the material element owing to flow. When the system is dissipative the entropy function cannot be derived using the constitutive equation in the same way as given in the case of the thermoelastic material. By assuming that E is a function of $e_{ij}, \dot{e}_{ij}$ and T and proceeding in same way as the thermoelastic material derivation shows there is no entropy function that is a state variable. When dissipation is present it is converted to heat and this must be reflected in the contributions to the first law. This may be accomplished in this case by splitting the stress into two parts, σD_{ij} and σS_{ij}. The stress , σD_{ij}, is determined from the part of the constitutive equation causing dissipation while the stress, σS_{ij}, is determined from the part of the constitutive equation contributing to the recoverable elastic strain energy as follows,

$$\sigma_{ij} = \sigma D_{ij} + \sigma S_{ij}$$

$$\sigma D_{ij} = \rho \cdot \overline{\lambda} \cdot \dot{e}_{kk} \cdot \delta_{ij} + 2 \cdot \rho \cdot \overline{\mu} \cdot \dot{e}_{ij}$$

$$\sigma S_{ij} = \rho \cdot \overleftrightarrow{C} \cdot \left(e_{kk} - 3 \cdot \alpha \cdot T\right) \cdot \delta_{ij}$$

The rate of work being done by the system is $-\frac{\sigma S_{ij} \cdot \dot{e}_{ij}}{\rho}$. Assume the internal energy, E, is a function of e_{kk} and T so that the first law gives,

$$\dot{q} = \frac{\partial E}{\partial e_{kk}} \cdot \dot{e}_{kk} + \frac{\partial E}{\partial T} \cdot \dot{T} - \frac{\sigma S_{ij} \cdot \dot{e}_{ij}}{\rho} = \left(\frac{\partial E}{\partial e_{kk}} \cdot \delta_{ij} - \frac{\sigma S_{ij}}{\rho} \right) \cdot \dot{e}_{ij} + \frac{\partial E}{\partial T} \cdot \dot{T}$$

and the entropy production rate, $\dot{s}$, is

$$\dot{s} = \frac{1}{T} \cdot \left(\frac{\partial E}{\partial e_{kk}} \cdot \delta_{ij} - \frac{\sigma S_{ij}}{\rho} \right) \cdot \dot{e}_{ij} + \frac{\partial E}{\partial T} \cdot \frac{\dot{T}}{T}$$

In order for the entropy to be a state property,

$$-\frac{1}{T^2} \cdot \left(\frac{\partial E}{\partial e_{kk}} \cdot \delta_{ij} - \frac{\sigma S_{ij}}{\rho} \right) + \frac{1}{T} \cdot \left(\frac{\partial^2 E}{\partial T \partial e_{kk}} \cdot \delta_{ij} - \frac{\partial}{\partial T} \left(\frac{\sigma S_{ij}}{\rho} \right) \right) = \frac{1}{T} \cdot \frac{\partial^2 E}{\partial e_{ij} \partial T} = \frac{1}{T} \cdot \frac{\partial^2 E}{\partial e_{kk} \partial T} \cdot \delta_{ij}$$

Assuming the order of differentiation for the second derivatives are interchangeable, the equation becomes,

$$T \cdot \frac{\partial}{\partial T} \left(\frac{\sigma S_{ij}}{\rho} \right) - \frac{\sigma S_{ij}}{\rho} = - \frac{\partial E}{\partial e_{kk}} \cdot \delta_{ij}$$

When the constitutive equation for σS_{ij} is substituted into the above equation, the result is,

$$\frac{\partial E}{\partial e_{kk}} = \vec{C} \cdot \left(e_{kk} + 3 \cdot \alpha \cdot T_O \right)$$

The last equation is integrated to give,

$$E = \vec{C} \cdot \left(\frac{1}{2} \cdot e_{kk}{}^2 + 3 \cdot \alpha \cdot T_O \cdot e_{kk} \right) + J(T)$$

where J(T) is an arbitrary function of T. When E is substituted into the expressions for heat flow rate and entropy production rate given above, the expressions become,

$$\dot{q} = 3 \cdot \alpha \cdot T \cdot \vec{C} \cdot \dot{e}_{kk} + \frac{d J(T)}{dT} \cdot \dot{T}$$

$$\dot{s} = 3 \cdot \alpha \cdot \vec{C} \cdot \dot{e}_{kk} + \frac{d J(T)}{dT} \cdot \frac{\dot{T}}{T}$$

When $\dot{e}_{kk} = 0$, the heat flow rate is usually written as $c_V \cdot \dot{T}$ with c_V being the specific heat. In this case,

$$\frac{dJ(T)}{dT} = c_V$$

When c_V is a constant, integration yields,

$$J(T) = c_V \cdot (T - T_1)$$

where T_1 is a constant of integration. To summarize,

$$\dot{q} = 3 \cdot \alpha \cdot T \cdot \vec{C} \cdot \dot{e}_{kk} + c_V \cdot \dot{T}$$

$$\dot{s} = 3 \cdot \alpha \cdot \vec{C} \cdot \dot{e}_{kk} + c_V \cdot \frac{\dot{T}}{T}$$

$$E = \tfrac{1}{2} \cdot \vec{E} \cdot e_{kk}{}^2 + 3 \cdot \vec{C} \cdot \alpha \cdot T_O \cdot e_{kk} + c_V \cdot (T - T_1)$$

The value of $\dot{q}$ in these equations is the total heat flow rate for the system and some is generated internally while the remainder is supplied externally to the material element. The internal heat flow rate, $\dot{q}_D$, is,

$$\dot{q}_D = \frac{\sigma D_{ij} \cdot \dot{e}_{ij}}{\rho} = \left(\vec{\lambda} \cdot \dot{e}_{kk} \cdot \delta_{ij} + 2 \cdot \vec{\mu} \cdot \dot{e}_{ij}\right) \dot{e}_{ij}$$

Now let the externally supplied heat flow rate be $\dot{\hat{q}}$ so that,

$$\dot{q} = \dot{\hat{q}} + \dot{q}_D$$

and,

$$\dot{\hat{q}} = 3 \cdot \alpha \cdot T \cdot \vec{C} \cdot \dot{e}_{kk} + c_V \cdot \dot{T} - \overline{\lambda} \cdot \dot{e}_{kk}{}^2 - 2 \cdot \overline{\mu} \cdot \dot{e}_{ij} \cdot \dot{e}_{ij}$$

ELASTIC, PERFECTLY-PLASTIC SOLID

The formulation investigated here is the one appearing in the text, *Theory of Perfectly Plastic Solids*, by William Prager and Philip Hodge, Jr. (John Wiley & Sons, Inc., 1951). The von Mises stress, σ_{VM}, is defined to be,

$$\sigma_{VM} = \sqrt{\tfrac{3}{2} \cdot \sigma_{ij} \cdot \sigma_{ij} - \tfrac{1}{2} \cdot \sigma_{kk}{}^2}$$

and possible stress states must be such that,

$$\sigma_{VM} \le \sigma_{YP}$$

where σ_{YP} is the yield point of the material, a constant. The strain is split into two parts. The elastic strain, eE_{ij}, is directly related to the stress state while the plastic strain, eP_{ij}, is adjusted to be proportional to the reduced stress, $\sigma_{ij} - \frac{1}{3} \cdot \sigma_{kk} \cdot \delta_{ij}$. The total strain rate is the sum of the elastic strain rate and the plastic strain rate, $\dot{e}E_{ij} + \dot{e}P_{ij}$. The relationship between the stress and the elastic strain is,

$$\sigma_{ij} = \left(\rho \cdot \breve{\lambda} \cdot eE_{kk} - \left(3 \cdot \rho \cdot \breve{\lambda} + 2 \cdot \rho \cdot \breve{G}\right) \alpha \cdot T\right) \delta_{ij} + 2 \cdot \rho \cdot \breve{G} \cdot eE_{ij}$$

where the material parameter nomenclature is the same nomenclature used for the elastic material considered above. The plastic strain changes over a loading increment when $\sigma_{VM} = \sigma_{YP}$ during the increment. This change is expressed by,

$$\dot{e}P_{ij} = \Gamma \cdot \left(\sigma_{ij} - \tfrac{1}{3} \cdot \sigma_{kk} \cdot \delta_{ij}\right) \qquad \text{while } \sigma_{VM} = \sigma_{YP}$$

$$\dot{e}P_{ij} = 0 \qquad \text{otherwise}$$

where Γ must be adjusted so that $\sigma_{VM} = \sigma_{YP}$. Note that $\dot{e}P_{kk} = 0$ so that the plastic strains cause no rate of volume change. The rate of doing external work for the elastic strain is assumed to be recoverable while the rate of doing work for the plastic strain is assumed to be dissipated into a heat flow rate within the material element. Define these as,

$$\dot{W}_E = -\frac{\sigma_{ij} \cdot \dot{e}E_{ij}}{\rho} = -\left(\breve{\lambda} \cdot eE_{kk} - \left(3 \cdot \breve{\lambda} + 2 \cdot \breve{G}\right) \alpha \cdot T\right) \dot{e}E_{kk} - 2 \cdot \breve{G} \cdot eE_{ij} \cdot \dot{e}E_{ij}$$

$$\dot{W}_P = -\frac{\sigma_{ij} \cdot \dot{e}P_{ij}}{\rho} = -2 \cdot \breve{G} \cdot eE_{ij} \cdot \Gamma \cdot \left(\sigma_{ij} - \tfrac{1}{3} \cdot \sigma_{kk} \cdot \delta_{ij}\right) \text{ while } \sigma_{VM} = \sigma_{YP}$$

$$\dot{W}_P = 0 \qquad \text{otherwise}$$

Now assume that the internal energy, E, is a function of the elastic strain, eE_{ij}, and the temperature T. For this material the first law is written as,

$$\dot{q} = \dot{E} + \dot{W}_E$$

so that,

$$\dot{q} = \left(\frac{\partial E}{\partial eE_{ij}} - \left(\breve{\lambda} \cdot eE_{kk} - \left(3 \cdot \breve{\lambda} + 2 \cdot \breve{G} \right) \alpha \cdot T \right) \delta_{ij} - 2 \cdot \breve{G} \cdot eE_{ij} \right) \cdot \dot{e}E_{ij} + \frac{\partial E}{\partial T} \cdot \dot{T}$$

and,

$$\dot{s} = \frac{1}{T} \cdot \left(\frac{\partial E}{\partial eE_{ij}} - \left(\breve{\lambda} \cdot eE_{kk} - \left(3 \cdot \breve{\lambda} + 2 \cdot \breve{G} \right) \alpha \cdot T \right) \delta_{ij} - 2 \cdot \breve{G} \cdot eE_{ij} \right) \cdot \dot{e}E_{ij} + \frac{1}{T} \cdot \frac{\partial E}{\partial T} \cdot \dot{T}$$

In order for the entropy to be a property,

$$-\frac{1}{T^2} \cdot \left(\frac{\partial E}{\partial eE_{ij}} - \left(\breve{\lambda} \cdot eE_{kk} - \left(3 \cdot \breve{\lambda} + 2 \cdot \breve{G} \right) \alpha \cdot T \right) \delta_{ij} - 2 \cdot \breve{G} \cdot eE_{ij} \right) + \frac{1}{T} \cdot \left(\frac{\partial^2 E}{\partial T \, \partial eE_{ij}} + \left(3 \cdot \breve{\lambda} + 2 \cdot \breve{G} \right) \alpha \cdot \delta_{ij} \right)$$
$$= \frac{1}{T} \cdot \frac{\partial^2 E}{\partial eE_{ij} \, \partial T}$$

As usual, the order of differentiation of the second derivatives is assumed interchangeable so that,

$$\frac{\partial E}{\partial eE_{ij}} = \breve{\lambda} \cdot \cdot eE_{kk} \cdot \delta_{ij} + 2 \cdot \breve{G} \cdot eE_{ij}$$

When this equation is integrated there results,

$$E = \tfrac{1}{2} \cdot \breve{\lambda} \cdot eE_{kk}{}^2 + \breve{G} \cdot eE_{ij} \cdot eE_{ij} + K(T)$$

and,

$$\dot{q} = \left(3 \cdot \breve{\lambda} + 2 \cdot \breve{G} \right) \alpha \cdot T \cdot \dot{e}E_{kk} + \frac{d\,K(T)}{dT} \cdot \dot{T}$$

$$\dot{s} = \left(3 \cdot \breve{\lambda} + 2 \cdot \breve{G} \right) \alpha \cdot \dot{e}E_{kk} + \frac{1}{T} \cdot \frac{\partial\,K(T)}{\partial T} \cdot \dot{T}$$

Now define a specific heat at constant volume, c_V, using,

$$\dot{q}\Big|_{\dot{e}E_{kk}} = c_V \cdot \dot{T}$$

to obtain,

$$c_V = \frac{d\,K(T)}{dT}$$

where, obviously, c_V is a function of temperature only. The heat flow rate and entropy production rate may be written as,

$$\dot{q} = \left(3 \cdot \breve{\lambda} + 2 \cdot \breve{G}\right) \alpha \cdot T \cdot \dot{e}E_{kk} + c_V \cdot \dot{T}$$

$$\dot{s} = \left(3 \cdot \breve{\lambda} + 2 \cdot \breve{G}\right) \alpha \cdot \dot{e}E_{kk} + \frac{1}{T} \cdot c_V \cdot \dot{T}$$

Similar to the case for a fluid, the quantity $\dot{q}$ is the total heat flow rate in the material element. The quantity $\dot{W}_P$ is the rate of work for the plastic strain that is converted to a heat flow rate. Let $\dot{\breve{q}}$ be the externally supplied heat flow rate so that,

$$\dot{\breve{q}} = \dot{q} - \dot{W}_P = \left(3 \cdot \breve{\lambda} + 2 \cdot \breve{G}\right) \alpha \cdot T \cdot \dot{e}E_{kk} + c_V \cdot \dot{T} + 4 \cdot \rho \cdot \breve{G}^2 \cdot \Gamma \cdot \left(eE_{ij} \cdot eE_{ij} - \tfrac{1}{3} \cdot eE_{kk}{}^2\right)$$

INCOMPRESSIBLE BINGHAM MATERIAL.

The most common formulation neglects thermal expansion and elastic behavior of the material and this approximation is employed here. This material has a yield point stress that must be exceeded before the material can deform. When the yield stress is exceeded, the material flows similar to a fluid but with the flow rate proportional to the excess of the stress over the yield point stress. Let,

$$\sigma_{VM} = \sqrt{\tfrac{3}{2} \cdot \sigma_{ij} \cdot \sigma_{ij} - \tfrac{1}{2} \cdot \sigma_{kk}{}^2} = \sqrt{\tfrac{3}{2}} \cdot \sqrt{\sigma_{ij} \cdot \sigma_{ij} - \tfrac{1}{3} \cdot \sigma_{kk}{}^2} = \sqrt{3} \cdot \tau_{VM}$$

where σ_{VM} is the von Mises stress in tension and τ_{VM} is the von Mises stress in shear. The yield point stress in shear is denoted by τ_{YP} and it is the value of the von Mises stress in simple shear that causes yielding of the material. For this material it is common to formulate the constitutive equation in terms of the constant value of τ_{YP}. The constitutive equation for the incompressible Bingham material is.

$$2 \cdot \mu \cdot \dot{e}_{ij} = \frac{\sigma_{VM} - \sqrt{3} \cdot \tau_{YP}}{\sigma_{VM}} \cdot \left(\sigma_{ij} - \tfrac{1}{3} \cdot \sigma_{kk} \cdot \delta_{ij}\right) \qquad \text{when } \sigma_{VM} \geq \sqrt{3} \cdot \tau_{YP}$$

$$2 \cdot \mu \cdot \dot{e}_{ij} = 0 \qquad \text{otherwise}$$

When this material undergoes deformation, the entire rate of work done by the stresses is converted to a heat flow rate. Consequently, in the first law $\dot{W} = 0$. In addition, the internal energy is assumed to be a function of temperature only. Under these conditions, the first law becomes,

$$\dot{q} = \frac{\partial E}{\partial T} \cdot \dot{T}$$

and the entropy production rate is,

$$\dot{s} = \frac{\partial E}{\partial T} \cdot \frac{\dot{T}}{T}$$

Clearly, $\frac{\partial E}{\partial T}$ may be interpreted as the specific heat at constant volume, c_V, so that,

$$\dot{q} = c_V \cdot \dot{T}$$

$$\dot{s} = c_V \cdot \frac{\dot{T}}{T}$$

During deformation, the quantity $-\frac{\sigma_{ij} \cdot \dot{e}_{ij}}{\rho}$ is the heat flow rate internal to the material element and,

$$\frac{\sigma_{ij} \cdot \dot{e}_{ij}}{\rho} = \frac{\sigma_{VM} - \sqrt{3} \cdot \tau_{YP}}{2 \cdot \rho \cdot \mu \cdot \sigma_{VM}} \cdot \left(\sigma_{ij} \cdot \sigma_{ij} - \tfrac{1}{3} \cdot \sigma_{kk}{}^2\right) = \frac{\left(\sigma_{VM} - \sqrt{3} \cdot \tau_{YP}\right)\sigma_{VM}}{3 \cdot \rho \cdot \mu} \quad , \ \sigma_{VM} \geq \sqrt{3} \cdot \tau_{YP}$$

so that the externally supplied heat flow rate, $\dot{\hat{q}}$, is,

$$\dot{\hat{q}} = c_V \cdot \dot{T} + \frac{\left(\sigma_{VM} - \sqrt{3} \cdot \tau_{YP}\right)\sigma_{VM}}{3 \cdot \rho \cdot \mu} \quad , \ \sigma_{VM} \geq \sqrt{3} \cdot \tau_{YP}$$

$$\dot{\hat{q}} = c_V \cdot \dot{T} \quad , \text{otherwise}$$

Owing to the assumption of incompressibility the mean stress, $\frac{1}{3} \cdot \sigma_{kk}$, is indeterminate from the deformation. A similar situation occurs in the case of any incompressible material..

PENG-ROBINSON CUBIC EQUATION OF STATE

This equation is used frequently to represent the state of the material in vapor-liquid equilibrium calculations. For a specified state (vapor or liquid) the equation contains three constants, R, a and b. It relates the pressure, p, to the specific volume, v, and temperature, T, as follows,

$$p = \frac{R \cdot T}{v - b} - \frac{a}{v^2 + 2 \cdot b \cdot v - b^2}$$

For this case assume the internal energy is a function of the specific volume and the temperature, E = E(v, T). The first law yields,

$$\dot{q} = \left(\frac{\partial E}{\partial v} + p \right) \cdot \dot{v} + \frac{\partial E}{\partial T} \cdot \dot{T}$$

and then,

$$\dot{s} = \frac{\dot{q}}{T} = \frac{1}{T} \cdot \left(\frac{\partial E}{\partial v} + p \right) \cdot \dot{v} + \frac{\partial E}{\partial T} \cdot \frac{\dot{T}}{T}$$

In order for the entropy to be a state variable,

$$\frac{-1}{T^2} \cdot \left(\frac{\partial E}{\partial v} + p \right) + \frac{1}{T} \cdot \left(\frac{\partial^2 E}{\partial T\, \partial v} + \frac{\partial p}{\partial T} \right) = \frac{1}{T} \cdot \frac{\partial^2 E}{\partial v\, \partial T}$$

Assuming the second derivatives are independent of the order of differentiation gives,

$$\frac{\partial E}{\partial v} + p - T \cdot \frac{\partial p}{\partial T} = 0$$

When p is eliminated from this equation using the equation of state, the result is,

$$\frac{\partial E}{\partial v} = \frac{a}{v^2 + 2 \cdot b \cdot v - b^2}$$

and integration gives,

$$E = \frac{a}{\sqrt{8 \cdot b^2}} \cdot \ln\left(\frac{2 \cdot v + 2 \cdot b - \sqrt{8 \cdot b^2}}{2 \cdot v + 2 \cdot b + \sqrt{8 \cdot b^2}} \right) + L(T)$$

so that

$$\dot{q} = \frac{R \cdot T}{v - b} \cdot \dot{v} + \frac{dL(T)}{dT} \cdot \dot{T}$$

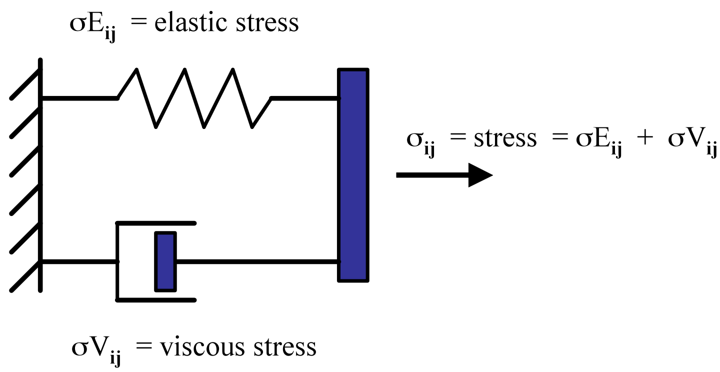
The multiplier of $\dot{T}$ is the specific heat at constant volume, c_V, so the heat flow rate and entropy production rate become,

$$\dot{q} = \frac{R \cdot T}{v - b} \cdot \dot{v} + c_V \cdot \dot{T}$$

$$\dot{s} = \frac{\dot{q}}{T} = \frac{R}{v - b} \cdot \dot{v} + c_V \cdot \frac{\dot{T}}{T}$$

VOIGT MATERIAL (KELVIN-VOIGT MATERIAL)

The sketch below is a conceptual description of this material. Although it is helpful to represent the physical characteristics of the material with this sort of sketch, D. C. Drucker (Second-order Effects in Elasticity. Plasticity and Fluid Dynamics, International Symposium, Haifa, Israel, April 23-27, 1962) has pointed out the limitations of such sketches.



The contributions to the stress, σE_{ij}, and σV_{ij}, are taken as the classical formulations for thermoelastic and viscous materials. The external work is associated with σE_{ij} only as the work associated with σV_{ij} is dissipated as heat in the material element. The total stress is the sum of the two contributions so,

$$\sigma E_{ij} = \rho \cdot \lambda \cdot (e_{kk} - 3 \cdot \alpha \cdot T) \cdot \delta_{ij} + 2 \cdot \rho \cdot G \cdot (e_{ij} - \alpha \cdot T \cdot \delta_{ij})$$

$$\sigma V_{ij} = \rho \cdot \overline{\lambda} \cdot \dot{e}_{kk} \cdot \delta_{ij} + 2 \cdot \rho \cdot \overline{\mu} \cdot \dot{e}_{ij}$$

$$\sigma_{ij} = \sigma E_{ij} + \sigma V_{ij}$$

The internal energy, E, is assumed to be a function of σE_{ij} and T only. The work term is taken as $-\dfrac{\sigma E_{ij} \cdot \dot{e}_{ij}}{\rho}$ and then the first law becomes.

$$\dot{q} = \left(\frac{\partial E}{\partial e_{ij}} - \frac{\sigma E_{ij}}{\rho} \right) \cdot \dot{e}_{ij} + \frac{\partial E}{\partial T} \cdot \dot{T}$$

and

$$\dot{s} = \frac{\dot{q}}{T} = \frac{1}{T}\cdot\left(\frac{\partial E}{\partial e_{ij}} - \frac{\sigma E_{ij}}{\rho}\right)\cdot\dot{e}_{ij} + \frac{\partial E}{\partial T}\cdot\frac{\dot{T}}{T}$$

The condition that must be satisfied in order that the entropy, s, be a state variable is,

$$\frac{-1}{T^2}\cdot\left(\frac{\partial E}{\partial e_{ij}} + \frac{\sigma E_{ij}}{\rho}\right) + \frac{1}{T}\cdot\left(\frac{\partial^2 E}{\partial T\,\partial e_{ij}} + \frac{\partial}{\partial T}\left(\frac{\sigma E_{ij}}{\rho}\right)\right) = \frac{1}{T}\cdot\frac{\partial^2 E}{\partial e_{ij}\,\partial T}$$

Assuming the order of differentiation for the second derivatives may be interchanged, this condition becomes,

$$\frac{\partial E}{\partial e_{ij}} = \frac{\sigma E_{ij}}{\rho} - T\cdot\frac{\partial}{\partial T}\left(\frac{\sigma E_{ij}}{\rho}\right)$$

When the constitutive equation is substituted into this condition, the result is,

$$\frac{\partial E}{\partial e_{ij}} = \lambda\cdot e_{kk}\cdot\delta_{ij} + 2\cdot G\cdot e_{ij}$$

and integration gives,

$$E = \tfrac{1}{2}\cdot\lambda\cdot e_{kk}{}^2 + G\cdot e_{ij}\cdot e_{ij} + M(T)$$

with M(T) being an arbitrary function of temperature. Recognizing that the specific heat at constant volume, c_V, is related to M(T) through,

$$\frac{d\,M(T)}{dT} = c_V$$

yields,

$$\dot{q} = (3\cdot\lambda + 2\cdot G)\cdot\alpha\cdot T\cdot\dot{e}_{kk} + c_V\cdot\dot{T}$$

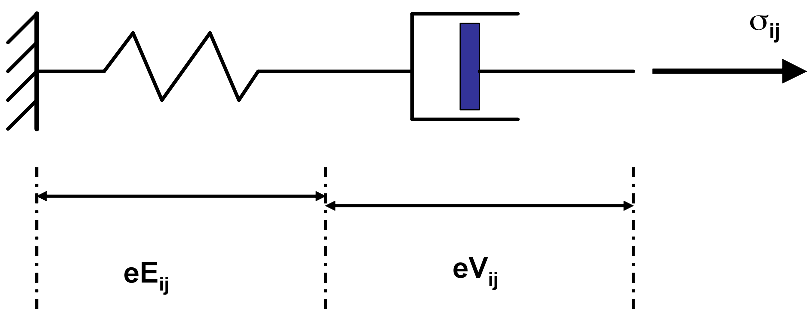
$$\dot{s} = (3\cdot\lambda + 2\cdot G)\cdot\alpha\cdot\dot{e}_{kk} + c_V\cdot\frac{\dot{T}}{T}$$

Since the internally generated heat flow rate is $-\dfrac{\sigma V_{ij}\cdot\dot{e}_{ij}}{\rho}$, the external heat flow rate, $\dot{\hat{q}}$, is given by,

$$\dot{\hat{q}} = \dot{q} + \frac{\sigma V_{ij} \cdot \dot{e}_{ij}}{\rho} = \dot{q} + \overline{\lambda} \cdot \dot{e}_{kk}{}^2 + 2 \cdot \overline{\mu} \cdot \dot{e}_{ij} \cdot \dot{e}_{ij}$$
$$= (3 \cdot \lambda + 2 \cdot G) \cdot \alpha \cdot T \cdot \dot{e}_{kk} + c_V \cdot \dot{T} + \overline{\lambda} \cdot \dot{e}_{kk}{}^2 + 2 \cdot \overline{\mu} \cdot \dot{e}_{ij} \cdot \dot{e}_{ij}$$

MAXWELL MATERIAL

The sketch below gives a conceptual, physical understanding of the Maxwell material. The strain has separate elastic and viscous components, eE_{ij} and eV_{ij}, that are induced by the total stress, σ_{ij}.



The analytical model developed here uses classical definitions to relate σ_{ij}, eE_{ij} and eV_{ij} as follows,

$$\sigma_{ij} = \rho \cdot \lambda \cdot (eE_{kk} - 3 \cdot \alpha \cdot T) \cdot \delta_{ij} + 2 \cdot \rho \cdot G \cdot (eE_{ij} - \alpha \cdot T \cdot \delta_{ij})$$

$$\sigma_{ij} = \rho \cdot \overline{\lambda} \cdot \dot{e}V_{kk} \cdot \delta_{ij} + 2 \cdot \rho \cdot \overline{\mu} \cdot \dot{e}V_{ij}$$

and the total strain rate, $\dot{e}_{ij}$, is defined as,

$$\dot{e}_{ij} = \dot{e}E_{ij} + \dot{e}V_{ij}$$

The external rate of work is $-\frac{\sigma_{ij} \cdot \dot{e}E_{ij}}{\rho}$ while the internal rate of work that is converted to heat flow rate is $-\frac{\sigma_{ij} \cdot \dot{e}V_{ij}}{\rho}$ and the internal energy, E, is assumed to be a function of eE_{ij} and T only. The first law is,

$$\dot{q} = \left(\frac{\partial E}{\partial eE_{ij}} - \frac{\sigma_{ij}}{\rho} \right) \cdot \dot{e}E_{ij} + \frac{\partial E}{\partial T} \cdot \dot{T}$$

and

$$\dot{s} = \frac{\dot{q}}{T} = \frac{1}{T} \cdot \left(\frac{\partial E}{\partial eE_{ij}} - \frac{\sigma_{ij}}{\rho} \right) \cdot \dot{e}E_{ij} + \frac{\partial E}{\partial T} \cdot \frac{\dot{T}}{T}$$

The condition that entropy be a state variable is,

$$-\frac{1}{T^2} \cdot \left(\frac{\partial E}{\partial eE_{ij}} - \frac{\sigma_{ij}}{\rho} \right) + \frac{1}{T} \cdot \left(\frac{\partial^2 E}{\partial T \, \partial eE_{ij}} - \frac{\partial}{\partial T} \left(\frac{\sigma_{ij}}{\rho} \right) \right) = \frac{1}{T} \cdot \frac{\partial^2 E}{\partial eE_{ij} \, \partial T}$$

With the assumption that the order of differentiation may be interchanged, this equation becomes,

$$\frac{\partial E}{\partial eE_{ij}} = \frac{\sigma_{ij}}{\rho} - \frac{\partial}{\partial T} \left(\frac{\sigma_{ij}}{\rho} \right)$$

Substitution of the constitutive equation into this equation gives,

$$\frac{\partial E}{\partial eE_{ij}} = \lambda \cdot eE_{kk} \cdot \delta_{ij} + 2 \cdot G \cdot eE_{ij}$$

and integration leads to,

$$E = \tfrac{1}{2} \cdot \lambda \cdot eE_{kk}{}^2 + G \cdot eE_{ij} \cdot eE_{ij} + N(T)$$

The function of integration, N(T), is related to the specific heat at constant volume, c_V, through,

$$c_V = \frac{d\,N(T)}{dT}$$

so that,

$$\dot{q} = (3 \cdot \lambda + 2 \cdot \mu) \cdot \alpha \cdot T \cdot \dot{e}E_{kk} + c_V \cdot \dot{T}$$

$$\dot{s} = (3 \cdot \lambda + 2 \cdot \mu) \cdot \alpha \cdot \dot{e}E_{kk} + c_V \cdot \frac{\dot{T}}{T}$$

The external heat flow rate, $\dot{\hat{q}}$, is the difference between the total heat flow rate, $\dot{q}$, and the internal heat flow rate, $-\dfrac{\sigma_{ij} \cdot \dot{e}E_{ij}}{\rho}$, so that,

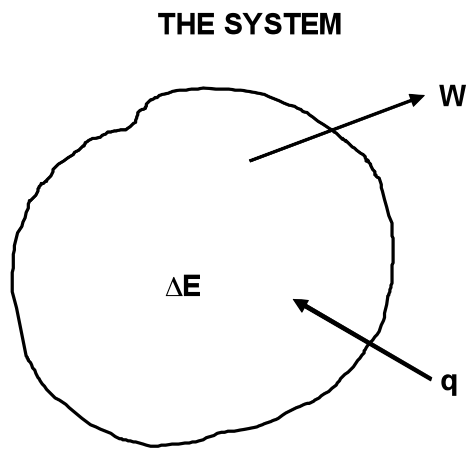
$$\dot{\hat{q}} = \dot{q} + \frac{\sigma_{ij} \cdot \dot{e}V_{ij}}{\rho} = \dot{q} + \overline{\lambda} \cdot \dot{e}V_{kk}{}^{2} + 2 \cdot \overline{\mu} \cdot \dot{e}V_{ij} \cdot \dot{e}V_{ij}$$

$$= \left(3 \cdot \lambda + 2 \cdot \mu\right) \cdot \alpha \cdot T \cdot \dot{e}E_{kk} + c_{V} \cdot \dot{T} + \overline{\lambda} \cdot \dot{e}V_{kk}{}^{2} + 2 \cdot \overline{\mu} \cdot \dot{e}V_{ij} \cdot \dot{e}V_{ij}$$

APPENDIX, CLASSICAL THERMODYNAMICS

FIRST LAW

Define a system as a fixed mass enclosed in a single surface boundary



ΔE = change in internal energy of system
W = work done by system
q = heat flow into system

The first law of Thermodynamics is a statement of conservation of energy for a system undergoing a thermodynamic change.

$$\Delta E = q - W$$

In classical thermodynamics the development is simplified by assuming the state of the system is dependent on only three variables that are related through an equation of state.

Variables and equation of state:

$$f(P, V, T) = 0$$

where,

P = pressure
V = volume
T = temperature, the exact scale to be used is defined later

Define the specific heat at constant V for the system as,

$$C_V = \frac{\partial E}{\partial T}\Big|_V$$

Define the specific heat at constant P for the system as,

$$C_P = \frac{\partial (E + P \cdot V)}{\partial T}\Big|_P$$

The internal energy may be considered as E(V, T) in view of the equation of state so that,

$$dE = \frac{\partial E}{\partial V}\Big|_T \cdot dV + \frac{\partial E}{\partial T}\Big|_V \cdot dT$$

Differentiating again with respect to T yields,

$$\frac{\partial E}{\partial T}\Big|_P = \frac{\partial E}{\partial V}\Big|_T \cdot \frac{\partial V}{\partial T}\Big|_P + \frac{\partial E}{\partial T}\Big|_V$$

so that,

$$C_P - C_V = \left(P + \frac{\partial E}{\partial V}\Big|_T\right) \cdot \frac{\partial V}{\partial T}\Big|_P$$

Define the enthalpy, H, as,

$$H \equiv E + P \cdot V$$

The Joule-Thompson coefficient, $\mu_{J.T.}$, is defined as,

$$\mu_{J.T.} = \frac{\partial T}{\partial P}\Big|_H$$

Ideal Gas

Temporarily consider the thermodynamic system to be a fixed mass of an ideal gas.

First part of definition of an ideal gas.

$$P \cdot V = R \cdot T\,, \quad R = 1.386 \frac{\text{Btu}}{(\text{lb}-\text{mol}) \cdot {}^{\circ}\text{F}}$$

An early experimental result by Joule is adopted as the other part of the definition of an ideal gas.

Joule's experiment $\rightarrow \frac{\partial E}{\partial V}\Big|_T = 0$

In this case,

$$dE = \frac{\partial E}{\partial V}\Big|_T \cdot dV + \frac{\partial E}{\partial T}\Big|_V \cdot dT = \frac{\partial E}{\partial T}\Big|_V \cdot dT$$

Therefore E = E(T) and

1 $$C_V = C_V(T),\ C_P - C_V = P \cdot \frac{\partial V}{\partial T}\Big|_P = R,\ C_P = C_P(T),\ \gamma = \frac{C_P}{C_V}$$

2. $$\Delta E = \int_{T_1}^{T_2} C_V \cdot dT \qquad \Delta H = \int_{T_1}^{T_2} C_P \cdot dT$$

3. For reversible isothermal process $q = W = R \cdot T \cdot \ln\left(\frac{V_2}{V_1}\right) = R \cdot T \cdot \ln\left(\frac{P_1}{P_2}\right)$

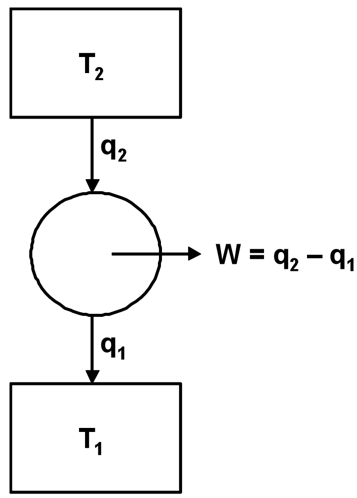
4. For reversible adiabatic process $W = -\int_{T_1}^{T_2} C_V \cdot dT \qquad C_V \cdot \frac{dT}{T} + R \cdot \frac{dV}{V} = 0$

If, in addition, C_V is constant $C_V \cdot \ln\left(\frac{T_2}{T_1}\right) + R \cdot \ln\left(\frac{V_2}{V_1}\right) = 0 \qquad P_1 \cdot V_1{}^{\gamma} = P_2 \cdot V_2{}^{\gamma}$

SECOND LAW

Return to considerations of the general case (not necessarily an ideal gas).

Carnot (reversible) engine



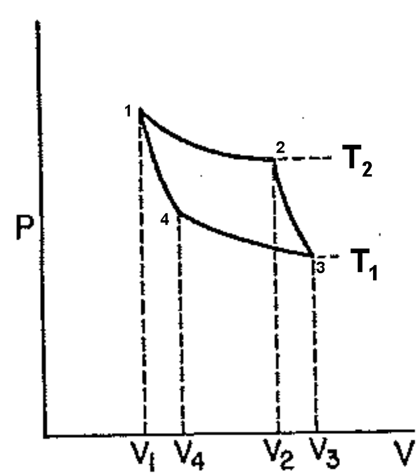
$$e = \text{efficiency} = \frac{W}{q_2} = \frac{q_2 - q_1}{q_2} = 1 - \frac{q_1}{q_2}$$

Consider two identical Carnot engines running in opposite directions. Unless they have same efficiency a perpetual motion can be built. Therefore,

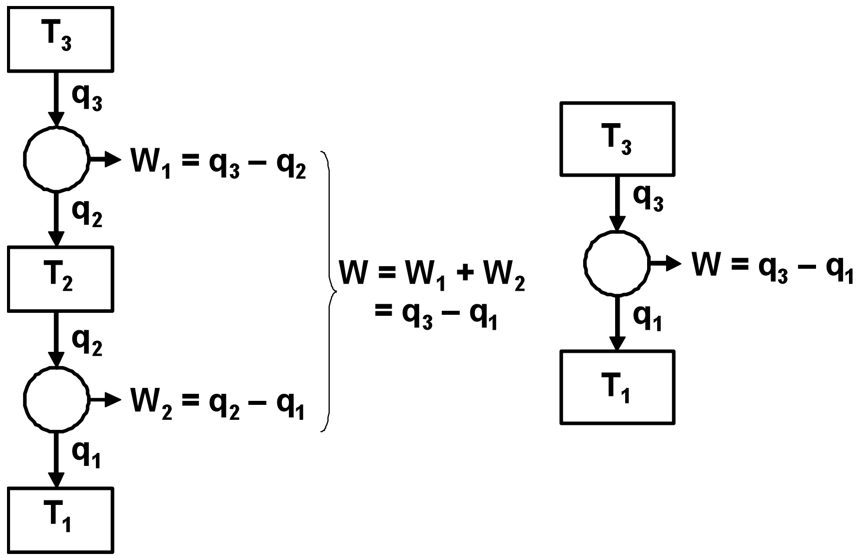
$$e = \bar{f}(T_1, T_2) \qquad \frac{q_1}{q_2} = f(T_1, T_2)$$

For a general gas running to produce W > 0

,



P_1, V_1 to P_2, V_2 is a reversible isothermal expansion at temperature T_2
P_2, V_2 to P_3, V_3 is a reversible adiabatic expansion to temperature T_1
P_3, V_3 to P_4, V_4 is a reversible isothermal compression at temperature T_1
P_4, V_4 to P_1, V_1 is a reversible adiabatic compression to temperature T_2



From left Carnot engines, From right Carnot engine,

$$\frac{q_2}{q_3} = f(T_2, T_3)$$

$$\frac{q_1}{q_2} = f(T_1, T_2) \qquad \frac{q_1}{q_3} = f(T_1, T_3)$$

$$\frac{q_1}{q_3} = f(T_1, T_2) \cdot f(T_2, T_3)$$

The two conitions are equivalent, therefore,

$f(T_1, T_3) = f(T_1, T_2) \cdot f(T_1, T_3) \rightarrow$ $$f(T_1, T_2) = \frac{f(T_1, T_3)}{f(T_2, T_3)} = \frac{q_1}{q_2}$$

Note that $f(T_1, T_2)$ is independent of T_3 so that $\rightarrow$ $$\frac{q_1}{q_2} = f(T_1, T_2) = \frac{F(T_1)}{G(T_2)}$$

Then $$\frac{F(T_1)}{G(T_3)} = \frac{F(T_1)}{G(T_2)} \cdot \frac{F(T_2)}{G(T_3)} \quad \rightarrow \quad G(T_2) = F(T_2)$$

Now choose $F(T) = T$ so that T is the thermodynamic temperature scale and,

$$\frac{q_1}{q_2} = f(T_1, T_2) = \frac{T_1}{T_2}$$

When the temperature is defined this way it is called Kelvin's thermodynamic temperature scale.

Before using the ideal gas law, it is necessary to check to see if the temperature in this law is consistent with the above definition of temperature. Therefore, return to the ideal gas law temporarily. Calculate W and q for each of the four parts of the Carnot cycle.

Isothermal expansion: $W_1 = q_2 = R \cdot T_2 \cdot \ln\left(\frac{V_2}{V_1}\right)$

Adiabatic expansion: $W_2 = \int_{T_1}^{T_2} C_V \cdot dT \qquad q = 0$

Isothermal compression: $W_3 = -q_1 = R \cdot T_1 \cdot \ln\left(\frac{V_4}{V_3}\right)$

Adiabatic compression: $W_4 = -\int_{T_1}^{T_2} C_V \cdot dT \qquad q = 0$

$$W = W_1 + W_2 + W_3 + W_4 = R \cdot T_2 \cdot \ln\left(\frac{V_2}{V_1}\right) + R \cdot T_1 \cdot \ln\left(\frac{V_4}{V_3}\right)$$

Using the thermodynamic temperature result, $\frac{q_1}{q_2} = \frac{T_1}{T_2}$, yields $\frac{V_2}{V_1} = \frac{V_3}{V_4}$ so that,

$W = R \cdot (T_2 - T_1) \cdot \ln\left(\frac{V_2}{V_1}\right)$ and then

$$e = \text{efficiency} = \frac{W}{q_2} = \frac{T_2 - T_1}{T_2}$$

This result for efficiency is the same as the earlier definition so the temperature in the ideal gas law is on a thermodynamic temperature scale.

Returning now to the general case, a common definition for the Second Law of Thermodynamics is that for any reversible engine.

$\oint \frac{dq}{T} \equiv \oint dS = 0$ where S is the entropy and for any engine that is not reversible,

$$\oint \frac{dq}{T} < 0 \qquad \text{e. g.} \quad \frac{q_2}{T_2} - \frac{q_1}{T_1} < 0$$

The above equation is known as the Inequality of Clausis.

Return again to the ideal gas to determine some expressions for entropy.

$$dq = dE + P \cdot dV = C_V \cdot dT + \frac{R \cdot T}{V} \cdot dV$$

$$dS = \frac{dq}{T} = C_V \cdot \frac{dT}{T} + R \cdot \frac{dV}{V}$$

If C_V is constant,

$$S_2 - S_1 = C_V \cdot \ln\left(\frac{T_2}{T_1}\right) + R \cdot \ln\left(\frac{V_2}{V_1}\right)$$

If V is constant also,

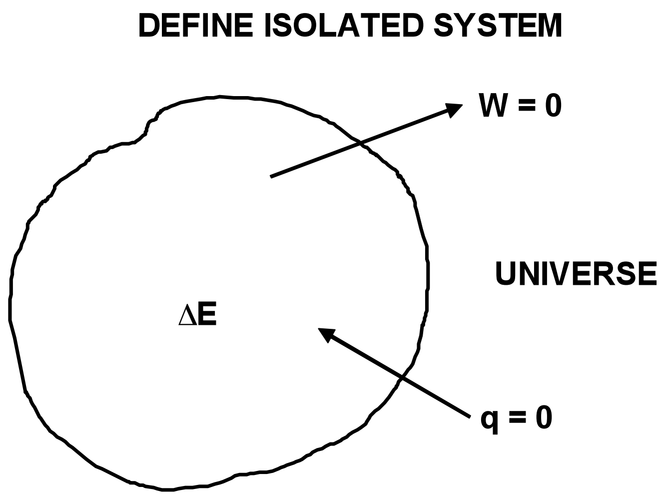
$$S_2 - S_1 = C_V \cdot \ln\left(\frac{T_2}{T_1}\right)$$

If, instead, the change is isothermal,

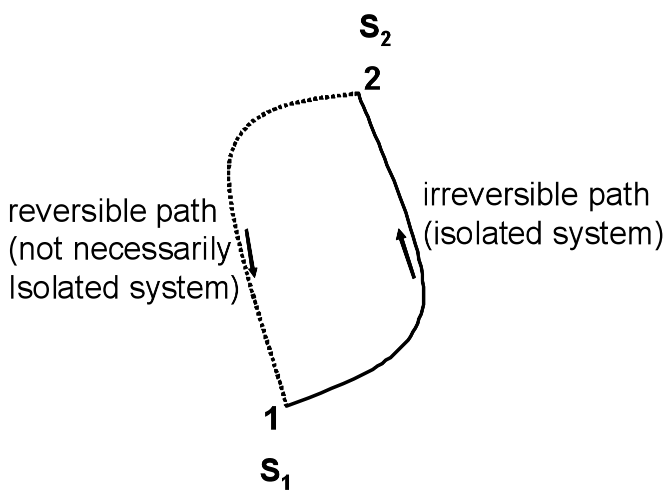
$$S_2 - S_1 = R \cdot \ln\left(\frac{V_2}{V_1}\right) = R \cdot \ln\left(\frac{P_1}{P_2}\right)$$

EQUILIBRIUM CONSIDERATIONS

First, an isolated system is defined as having W = 0 and q = 0 as shown below.



Consider the following cycle for this isolated system.



For this cycle, recalling the Inequality of Clausis,

$$\int_{S_1}^{S_2} \frac{dq}{T}\bigg|_{\text{IRREVERSIBLE}} + \int_{S_2}^{S_1} \frac{dq}{T}\bigg|_{\text{REVERSIBLE}} < 0$$

The first integral must vanish since q = 0. The second integral equals $S_1 - S_2$. Therefore,

$S_1 - S_2 < 0$ or $S_2 - S_1 > 0$

This shows that, whereas the energy of the universe is constant, the entropy of the universe is approaching a maximum. An isolated system may also be defined as having E and V constant.

Consider a spontaneous change in an isolated system. It must be accompanied by $\Delta S > 0$.

Equilibrium is defined as the state where no spontaneous changes occur. From this two equivalent equilibrium criteria are deduced. They are,

1 At constant E and V the entropy is maximized.
2 At constant S and V the internal energy is minimized.

Although these are valid, they have limited use. The second criterion is applied for spring-mass systems in mechanics thus leading to the minimum energy theorem.

The above two equilibrium conditions are not too useful in chemistry. Now get two more equilibrium related results that are widely used. Let,

A = work function or Helmholtz free energy = $E - T \cdot S$

F = thermodynamic potential = free energy = Gibbs free energy = $H - T \cdot S$

$$F = A + P \cdot V$$

For a constant T reversible change,

$$\Delta A = \Delta E - T \cdot \Delta S \equiv -W_{MAX}$$

For a real system,

$$W < W_{MAX}$$

For a constant P reversible change,

$$\Delta F = \Delta A + P \cdot \Delta V$$

if this change is also a constant temperature change,

$$\Delta F = -W_{MAX} + P \cdot \Delta V \equiv -W_{NET}$$

Most laboratory experiments (electricity excluded) in chemistry are performed under conditions of constant T and P such that $W_{NET} = 0$ so that $\Delta F = 0$. Since $\Delta S > 0$ or $\Delta H < 0$ cause $\Delta F < 0$ another equilibrium condition is determined.

The two new equilibrium conditions are,

1. At constant T and P: F at equilibrium is a minimum
2. At constant T and V: A at equilibrium is a maximum

DETERMINATION OF PROPERTIES FROM EXPERIMENTAL RESULTS

$$dF = dE + P \cdot dV + V \cdot dP - T \cdot dS - S \cdot dT$$

$$dE = T \cdot dS - P \cdot dV$$

so that,

$$dF = V \cdot dP - S \cdot dT$$

$$\therefore \quad \frac{\partial F}{\partial P}\Big|_T = V \quad \text{and} \quad \frac{\partial F}{\partial T}\Big|_P = -S$$

For an isothermal change, $\Delta F = F_2 - F_1 = \int_1^2 V \cdot dP$ so , given an equation of state, if F is known at one pressure it can be found for any other pressure. For the case of an ideal gas, $\Delta F = R \cdot T \cdot \ln\left(\frac{P_2}{P_1}\right)$

For a change at constant pressure

$$\frac{\partial F}{\partial T}\Big|_P = -S = \frac{F - H}{T}$$

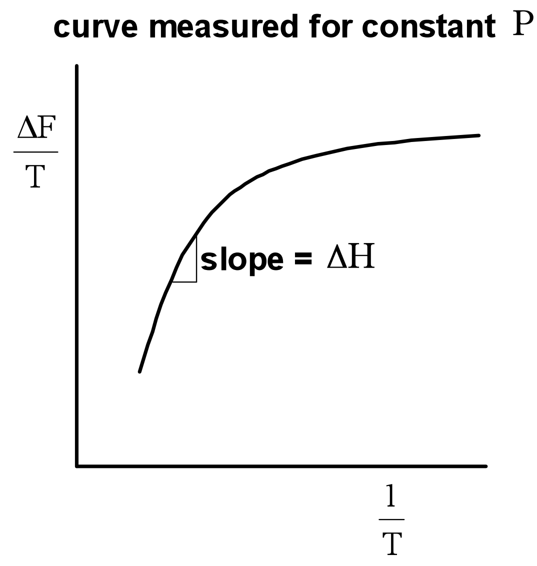
if, in addition, the change is at constant temperature,

$$\frac{\partial \Delta F}{\partial T}\Big|_P = -\Delta S = \frac{\Delta F - \Delta H}{T}$$

this equation is called the Gibbs-Helmholtz equation and it may be converted to the form,

$$\frac{\partial\left(\frac{\Delta F}{T}\right)}{\partial\left(\frac{1}{T}\right)}\Bigg|_P = \Delta H$$

Thus the slope of the plot of $\frac{\Delta F}{T}$ versus $\frac{1}{T}$ is equal to ΔH as shown below.



Other relations between the thermodynamic variables can be derived using the Gibbs free energy function, F. Consider the identity,

$$\frac{\partial}{\partial P}\left(\frac{\partial F}{\partial T}\right)\Big|_P = \frac{\partial}{\partial T}\left(\frac{\partial F}{\partial P}\right)\Big|_T$$

Since it has been shown that $\frac{\partial F}{\partial P}\Big|_T = V$ and $\frac{\partial F}{\partial T}\Big|_P = -S$ there results that,

$$\frac{\partial S}{\partial P}\Big|_T = -\frac{\partial V}{\partial T}\Big|_P$$

so at constant temperature,

$$\Delta S = -\int_{P_1}^{P_2} \frac{\partial V}{\partial T}\Big|_P \cdot dP = -\int_{P_1}^{P_2} \alpha \cdot V_O \cdot dP$$

This integration can be performed if the equation of state is known. In the case of an ideal gas we already have shown that $\Delta S = R \cdot \ln\left(\frac{V_2}{V_1}\right) = R \cdot \ln\left(\frac{P_1}{P_2}\right)$

At constant pressure,

$$dS = \frac{dq}{T} = \frac{dH}{T} = \frac{C_P \cdot dT}{T} \qquad \rightarrow \qquad \Delta S = \int_{T_1}^{T_2} \frac{C_P}{T} \cdot dT$$

At constant volume,

$$dS = \frac{dq}{T} = \frac{dE}{T} = \frac{C_V \cdot dT}{T} \qquad \rightarrow \qquad \Delta S = \int_{T_1}^{T_2} \frac{C_V}{T} \cdot dT$$

SUMMARY OF THERMODYNAMIC VARIABLES

$H = E + P \cdot V$	Enthalpy
$A = E - T \cdot S$	Helmholtz free energy or work function
$F = E + P \cdot V - T \cdot S$	Gibbs free energy or thermodynamic potential

$dE = T \cdot dS - P \cdot dV$
$dH = T \cdot dS + V \cdot dP$
$dA = -S \cdot dT - P \cdot dV$
$dF = -S \cdot dT + V \cdot dP$

$$\frac{\partial S}{\partial V}\Big|_T = \frac{\partial P}{\partial T}\Big|_V$$

$$\frac{\partial S}{\partial P}\Big|_T = \frac{\partial V}{\partial T}\Big|_P$$

$$C_P = \frac{dq}{dT}\Big|_P = T \cdot \frac{\partial S}{\partial T}\Big|_P$$

$$C_V = \frac{dq}{dT}\Big|_V = T \cdot \frac{\partial S}{\partial T}\Big|_V$$

$$\frac{\partial E}{\partial V}\Big|_T + P = T \cdot \frac{\partial P}{\partial T}\Big|_V$$

$$\frac{\partial H}{\partial P}\Big|_T - V = -T \cdot \frac{\partial V}{\partial T}\Big|_P$$

$$\mu_{J.T.} = \frac{\partial T}{\partial P}\Big|_H = -\frac{1}{C_P} \cdot \frac{\partial H}{\partial P}\Big|_T = \frac{T \cdot \frac{\partial V}{\partial T}\Big|_P - V}{C_P}$$

$$\alpha = \frac{1}{V_O} \cdot \frac{\partial V}{\partial T}\Big|_P$$