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THE FORM ALGORITHM AS IT IS APPLIED IN PRP REPORTS
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FORMULATION

The following notation is used in this presentation,

$\mathbf{t} = t + \boldsymbol{\delta}\mathbf{t}$

where,

$\mathbf{t}$ = a variable including its random variation governed by a normal distribution

t = value of $\mathbf{t}$ without its random variation, its mean time

$\boldsymbol{\delta}\mathbf{t}$ = variable part of $\mathbf{t}$ with zero mean and standard deviation of SDt

Consider the following nominal functional relationship,

$\mathbf{F} = \mathbf{F}(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N)$

where $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N$ are specified. The function $\mathbf{F}$ may be any relation (such as the dependence of the yield stress on the hardness). Therefore, the function itself may contain its own uncertainty. Define the variation in $\mathbf{F}$ owing only to the functional relationship (not the variations of its arguments) as $\boldsymbol{\Delta}\mathbf{F}$. Assume that $\boldsymbol{\Delta}\mathbf{F}$ in normally distributed with zero mean. Introducing the uncertainty into the above nominal relationship yields,

$\mathbf{F} = (1 + \boldsymbol{\Delta}\mathbf{F}) \cdot \mathbf{F}(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N)$

The problem considered in this appendix is the determination of the standard deviation of $\mathbf{F}$ given the functional relationship, $\boldsymbol{\Delta}\mathbf{F}$ and $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N$. With the above definitions the value of $\mathbf{F}$, F, at $x_1, x_2, \ldots, x_N$ given by evaluation of the nominal relationship is given by,

$F = \mathbf{F}(x_1, x_2, \ldots, x_N)$

and the standard deviation of $F \cdot \boldsymbol{\Delta}\mathbf{F}$ is defined as SDΔF

Now, to first order terms in the variable parts of $\mathbf{F}, \mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N$,

$$\mathbf{F} = F + \boldsymbol{\Delta}\mathbf{F} \cdot F + \sum_{i=1}^{N} \frac{\partial F}{\partial x_i} \cdot \boldsymbol{\delta}\mathbf{x}_i$$

or

$$\mathbf{F} - \mathrm{F} = \mathbf{\Delta F} \cdot \mathrm{F} + \sum_{i=1}^{N} \frac{\partial \mathrm{F}}{\partial x_i} \cdot \mathbf{\delta x_i}$$

The left hand side of the last equation is the variation of **F** or **δF** so that,

$$\mathbf{\delta F} = \mathrm{F} \cdot \mathbf{\Delta F} + \sum_{i=1}^{N} \frac{\partial \mathrm{F}}{\partial x_i} \cdot \mathbf{\delta x_i}$$

Assuming that **ΔF**, $\mathbf{\delta x_1}$, $\mathbf{\delta x_2}$, …, $\mathbf{\delta x_N}$ are independent random variables, it is straightforward to determine the standard deviation of **δF.** Virtually every statistics and probability textbook (e.g.- Sheldon M. Ross, *Introduction to Probability and Statistics for Engineers and Scientists*, 2nd Edition, Academic Press, 2000, pp. 118-123) gives the result that for this case of a sum of independent random variables the desired variance equals the sum of the variances. This leads to the equation that determines the standard deviation of **F**, SDF,

$$\mathrm{SDF}^2 = \mathrm{SD\Delta F}^2 + \sum_{i=1}^{N} \left(\frac{\partial \mathrm{F}}{\partial x_i} \cdot \mathrm{SDx}_i \right)^2$$

Occasionally, the above formulation is expressed in terms of coefficients of variation (COV). Introduce the following definitions for the pertinent coefficients of variation,

$$\mathrm{COVF} \equiv \frac{\mathrm{SDF}}{\mathrm{F}}$$

$$\mathrm{COV\Delta F} \equiv \frac{\mathrm{SD\Delta F}}{\mathrm{F}}$$

$$\mathrm{COVx}_i \equiv \frac{\mathrm{SDx}_i}{x_i} \qquad i = 1.2, \ldots, N$$

The equation corresponding to the above equation for SDF^2 becomes,

$$\mathrm{COVF}^2 = \mathrm{COV\Delta F}^2 + \sum_{i=1}^{N} \left(\frac{x_i}{\mathrm{F}} \cdot \frac{\partial \mathrm{F}}{\partial x_i} \cdot \mathrm{COVx}_i \right)^2$$

The equations for SDF^2 and COVF^2 show that, in general, the values of SDF and COVF depend on the values of x_i.

The above procedure may be followed for more complex functions. For example,

$$\mathbf{F} = \mathbf{F}(\mathbf{x_1}, \mathbf{x_2}, \ldots, \mathbf{x_N}, \mathbf{G}(\mathbf{y_1}, \mathbf{y_2}, \ldots, \mathbf{y_M}))$$

may be written as,

$$\mathbf{F = (1 + \Delta F) \cdot F(x_1, x_2, \ldots, x_N, (1 + \Delta G) \cdot G(y_1, y_2, \ldots, y_M))}$$

and this leads to,

$$\boldsymbol{\delta}\mathbf{F} = F \cdot \boldsymbol{\Delta}\mathbf{F} + \sum_{\mathbf{i=1}}^{\mathbf{N}} \frac{\partial F}{\partial x_i} \cdot \boldsymbol{\delta}\mathbf{x_i} + \frac{\partial F}{\partial G} \cdot \boldsymbol{\Delta}\mathbf{G} + \frac{\partial F}{\partial G} \cdot \sum_{\mathbf{i=1}}^{\mathbf{M}} \frac{\partial G}{\partial y_i} \cdot \boldsymbol{\delta}\mathbf{y_i}$$

so that,

$$SDF^2 = SD\Delta F^2 + \sum_{i=1}^{N} \left(\frac{\partial F}{\partial x_i} \cdot SDx_i \right)^2 + \left(\frac{\partial F}{\partial G} \cdot SD\Delta G \right)^2 + \sum_{i=1}^{M} \left(\frac{\partial F}{\partial G} \cdot \frac{\partial G}{\partial y_i} \cdot SDy_i \right)^2$$

and,

$$COVF^2 = COV\Delta F^2 + \sum_{i=1}^{N} \left(\frac{x_i}{F} \cdot \frac{\partial F}{\partial x_i} \cdot COVx_i \right)^2 + \left(\frac{G}{F} \cdot \frac{\partial F}{\partial G} \cdot COV\Delta G \right)^2 + \sum_{i=1}^{M} \left(\frac{G}{F} \cdot \frac{\partial F}{\partial G} \cdot \frac{\partial G}{\partial y_i} \cdot COVy_i \right)^2$$

Likewise,

$$\mathbf{F = H(x_1, x_2, \ldots, x_N) \cdot G(y_1, y_2, \ldots, y_M)}$$

may be written as,

$$\mathbf{F = (1 + \Delta H) \cdot H(x_1, x_2, \ldots, x_N) \cdot (1 + \Delta G) \cdot G(y_1, y_2, \ldots, y_M)}$$

and through first order terms this leads to,

$$\boldsymbol{\delta}\mathbf{F} = H \cdot G \cdot (\boldsymbol{\Delta}\mathbf{H} + \boldsymbol{\Delta}\mathbf{G}) + G \cdot \sum_{\mathbf{i=1}}^{\mathbf{N}} \frac{\partial H}{\partial x_i} \cdot \boldsymbol{\delta}\mathbf{x_i} + H \cdot \sum_{\mathbf{i=1}}^{\mathbf{M}} \frac{\partial G}{\partial y_i} \cdot \boldsymbol{\delta}\mathbf{y_i}$$

so that,

$$SDF^2 = (H \cdot SD\Delta G)^2 + (G \cdot SD\Delta H)^2 + \sum_{i=1}^{N} \left(G \cdot \frac{\partial H}{\partial x_i} \cdot SDx_i \right)^2 + \sum_{i=1}^{M} \left(H \cdot \frac{\partial G}{\partial x_i} \cdot SDy_i \right)^2$$

and

$$COVF^2 = COV\Delta G^2 + COV\Delta H^2 + \sum_{i=1}^{N} \left(\frac{x_i}{H} \cdot \frac{\partial H}{\partial x_i} \cdot COVx_i \right)^2 + \sum_{i=1}^{M} \left(\frac{y_i}{G} \cdot \frac{\partial G}{\partial y_i} \cdot COVy_i \right)^2$$

ILLUSTRATIVE EXAMPLE

Consider a set of three equations. The equations are,

$$\boldsymbol{\sigma}_{\mathbf{Y}} = (1 + \boldsymbol{\Delta\sigma}_{\mathbf{Y}}) \cdot (a_0 + a_1 \cdot \mathbf{HRC})$$

$$\mathbf{RAT} = (1 + \mathbf{\Delta RAT}) \cdot \left(b_0 + b_1 \cdot \mathbf{HRC} + b_2 \cdot \mathbf{HRC}^2 + b_3 \cdot \mathbf{HRC}^3\right)$$

$$\mathbf{SVM} = \boldsymbol{\sigma}_{\mathbf{Y}} \cdot \mathrm{fact} \cdot (\mathbf{RAT} + \mathrm{facs})$$

where the notation is the same as the above section (bold for random variables, regular and lower case for constants, regular for constant part of random variable and model variations begin with **Δ**) and

HRC = Rockwell C scale hardness, known random variable
$\boldsymbol{\sigma}_{\mathbf{Y}}$ = yield stress, random variable
RAT = threshold for stress ratio, random variable
SVM = allowable value for equivalent stress, random variable
$\boldsymbol{\Delta\sigma}_{\mathbf{Y}}$ = random variation in σ_Y(HRC) model with mean equal to zero, known
ΔRAT = random variation in RAT(HRC) model with mean equal to zero, known
a_0, a_1, b_0, b_1, b_2, b_3, fact, and facs are known constants

Clearly, there are two equations whose independent variable is **HRC**. The objective here is to determine the standard deviation of **SVM**, SDSVM, for a specified **HRC.** This will be accomplished by finding the standard deviation of $\boldsymbol{\sigma}_{\mathbf{Y}}$, $SD\sigma_Y$, and the standard deviation of **RAT**, SDRAT, from the first two equations and then using these results with the third equation to find SDSVM.

For the first equation, the first order variation of the random variables become,

$$\boldsymbol{\delta\sigma}_{\mathbf{Y}} = \boldsymbol{\delta\Delta\sigma}_{\mathbf{Y}} \cdot (a_0 + a_1 \cdot \mathrm{HRC}) + a_1 \cdot \boldsymbol{\delta}\mathbf{HRC}$$

Now applying the theorem concerning addition of independent random variables yields,

$$SD\sigma_Y{}^2 = (a_0 + a_1 \cdot \mathrm{HRC})^2 \cdot SD\Delta\sigma_Y{}^2 + a_1{}^2 \cdot SDHRC^2$$

so that $SD\sigma_Y$ is determined. For the second equation, the first order variation of the random variables gives,

$$\begin{aligned}\boldsymbol{\delta}\mathbf{RAT} = \boldsymbol{\delta\Delta}\mathbf{RAT} &\cdot \left(b_0 + b_1 \cdot \mathrm{HRC} + b_2 \cdot \mathrm{HRC}^2 + b_3 \cdot \mathrm{HRC}^3\right) \\ &+ \left(b_1 + 2 \cdot b_2 \cdot \mathrm{HRC} + 3 \cdot b_3 \cdot \mathrm{HRC}^2\right) \boldsymbol{\delta}\mathbf{HRC}\end{aligned}$$

Now applying the theorem concerning addition of independent random variables yields,

$$\begin{aligned}\mathrm{SDRAT}^2 &= \mathrm{SD\Delta RAT}^2 \cdot \left(b_0 + b_1 \cdot \mathrm{HRC} + b_2 \cdot \mathrm{HRC}^2 + b_3 \cdot \mathrm{HRC}^3\right)^2 \\ &\quad + \left(b_1 + 2 \cdot b_2 \cdot \mathrm{HRC} + 3 \cdot b_3 \cdot \mathrm{HRC}^2\right)^2 \cdot \mathrm{SDHRC}^2\end{aligned}$$

so that SDRAT is determined. The variable part of the third equations through linear terms becomes,

$$\boldsymbol{\delta}\mathbf{SVM} = \boldsymbol{\delta\sigma}_{\mathbf{Y}} \cdot \mathrm{fact} \cdot (\mathrm{RAT} + \mathrm{facs}) + \sigma_Y \cdot \mathrm{fact} \cdot \boldsymbol{\delta}\mathbf{RAT}$$

Now applying the theorem concerning addition of independent random variables yields,

$$\mathrm{SDSVM}^2 = \mathrm{SD}\sigma_Y{}^2 \cdot \mathrm{fact}^2 \cdot (\mathrm{RAT} + \mathrm{facs})^2 + \sigma_Y{}^2 \cdot \mathrm{fact}^2 \cdot \mathrm{SDRAT}^2$$

After SDσ_Y and SDRAT have been determined, the above equation determines SDSVM.

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